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ABSTRACT

This textbook was developed by the Colorado Schools Computing Science (CSCS) Curriculum Development Project. It can be used with high school or college students in an integrated presentation of second-year algebra, trigonometry, and beginning computer programing. (MK)

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A COURSE IN ALGEBRA AND TRIGONOMETRY WITH COMPUTER PROGRAMMING

A National Science Foundation
Curriculum Development Project

The University of Colorado

and

These Cooperating Colorado School Districts:

Boulder Valley, Cherry Creek, and Jefferson County

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Preface

This material was produced by the Colorado Schools Computing Science Group. The group consists of five teachers from the Jefferson County Public Schools, one teacher from the Boulder Valley Public Schools, one teacher from the Cherry Creek Public Schools, and several faculty members from the University of Colorado. Financial support was provided through the University of Colorado by a grant from the National Science Foundation.

The Colorado Schools Computing Science Group was formed to study the uses of computers in the high school and college mathematics curriculum. Specifically, the group sought answers to these basic questions:

1. Can the computer be used to improve the high school and college student's understanding of mathematics?

2. How will the emergence of the computer affect the high school and college mathematics curriculum?

In order to gain the experience necessary to answer these questions, the group chose to develop a second year course in algebra and trigonometry with a completely integrated use of the digital computer. Second year algebra was chosen because the topics taught in this course give rise to meaningful illustrations of computer techniques as well as useful applications of these techniques.

Since the goal was to determine how the computer could be used effectively in the high school and college mathematics curriculum, rather than to develop a computer course, the computer concepts have been integrated directly with the mathematics throughout the text. This integration allows the student to study new computer techniques in the context of familiar mathematics, and conversely new mathematical ideas are introduced with computer techniques. Thus, each reenforces the other yet the result is a course in mathematics rather than computer science. No attempt is made to develop exceptionally proficient programmers, but rather to introduce computer concepts and techniques that will strengthen the understanding of mathematics. For example, the topic of programming is dealt with in a general manner in the context of the mathematics being studied.

The following objectives served as a guideline in the writing of the materials.

A. To make an integrated presentation of second year algebra, trigonometry, and beginning computer programming.

B. To present topics in a mathematically precise and pedagogically sound manner.

C. To use computer techniques to support and reinforce the presentation of the mathematics.

D. To use computer methods to stimulate student interest in mathematics.

E. To provide new learning experiences and teaching techniques which enhance the students understanding of mathematics such as:

1. Flowcharts
2. Student programming
3. Developing algorithms
4. Student modeling of mathematics concepts
5. Student use of library programs

F. To develop student understanding of the power and limitations of the computer.

Since one of the objectives was to make the power of the computer as readily available to the students as possible, BASIC language was chosen as the computer language to be used in this text. BASIC is easy to learn and at the same time it's capabilities are sufficient for the problems that will be confronted in the course.

Every attempt has been made to use the most widley accepted form of BASIC in the text. However, slight variations in this form of BASIC may exist on any given computer system. For example, the authors found several variations in the form of the calculated jump instruction. The teacher should always be ready to make adjustments to meet the peculiarities of the system being used. In addition, no attempt has been made in the text to introduce systems commands such as RUN, LIST, SAVE, CATALOGUE, etc. These commands should be introduced as students need them using the systems manual for the particular system involved as a guide.

The text does not present a study of hardware or the electronic components of a computer. We have developed only a method where the computer can be used to enhance students understanding of mathematics. The authors feel that a study of hardware and machine language programming should be avoided in this course. Teachers who feel a personal need for more information concerning hardware are directed to the bibliography of supplementary materials at the end of this introduction.

Most of the topics of the usual algebra-trigonometry course will be found in this text. However, the emphasis placed on various topics in this text may be different from that normally given in other texts. The authors feel this is a result of an honest effort to uncover the proper role of the computer in mathematics education. In Chapters 1-3, properties of sets, axioms of the real number system, rules of proof, and programming are presented. These concepts are used in a precise development of the mathematics in the remaining chapters. The concept of a function becomes the unifying thread of Chapters 4-10.

The first five chapters are basic to the course and should be taught in sequence. The order of the later chapters can be changed without difficulty.

It is intended that the materials be followed rather closely. The students should be required to do all EXERCISES. Problems assigned from the PROBLEM SETS should be chosen carefully to meet the needs of each particular student. REVIEW PROBLEM SETS contain problems which review the concepts learned in Algebra I or studied in earlier chapters of this text. Each teacher will have to decide the extent to which the review sets should be used, basing it upon the needs of the class.

The numbering of sections of the text is done in the usual manner, m - n, where m is the chapter number and n is the number of the section within that chapter. All Exercises, Problem Sets, Review Problem Sets, Examples and Figures are numbered successively in the order in which they appear in each section of the chapter. For instance, the number 3-7-2 indicates that we are referring to the second item in section 7 of chapter 3.

Several problems of management can occur due to the amount of computer time and money available. By grouping students and assigning a problem to the group the total amount of computer time used "on line" can be cut to a more reasonable amount than when students work individually. Grouping also means that only one tape must be punched or one deck of cards marked. Queue length is also cut down, eliminating impatience on the part of the student. Caution must be taken so that some members of the group will not do all the work while others sit quietly by. Certain problems lend themselves to larger groups of 5-6 students while for others, groups of 2 or 3 students are appropriate. The usual principles of group work should be employed where applicable.

One thing is imperative. Early in the year, each student must have a successful experience programming the computer. This will help to eliminate problems of student disinterest later. The teacher has an obligation to structure groups and individual assignments in such a way that is accomplished.

The class should be paced so that at least Chapters 1-5 are completed by the end of the first semester. The last five chapters might be more than some classes can complete in one semester. Therefore, the teacher will need to choose which topics are to be omitted.

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ADDENDA & ERRATA

Page 1-42: First line should be Example 1-6-11.

Page 1-52: "No" should be indicated as an output of the diamond shaped command.

Page 2-33: A minus sign should be added in Problem 18.

Page 3-2: Lines 19 and 20, the word "meaningless" should be replaced by "indeterminate".

Page 3-9: Following line 12, adding "By the Replacement Principle Va, $5a + 9 = 13 - 2a \leftrightarrow 7a + 9 = 13$.

Page 3-30: Draw a bold arrow from $-\frac{1}{3}$ toward -2.

Page 3-32: Line 9: (5) $(a > -6$ and $a > 1)$

Page 3-35: Problem 13, delete the last handwriting $\left|\frac{x}{y}\right|$.

Page 3-36: First line: it follows . . . . .
Third line: 1. Write proofs of the other true generalizations in Exercise 3-7-24 . . .

Page 3-45: Delete an inked star on the top left corner.

Page 3-46: Second line: first word should be "mixed".

Page 4-4: Exercise 4-2-7, 3,c. second term should be (-2,-2).

Page 4-6: Third line from the bottom, $0 \leq x \leq 3$.

Page 4-8: Figure 4-3-3, the numbers on the positive y-axis should be +1,+2.

Page 4-12: Between the 4th and 5th lines from the bottom, adding "Therefore we shade half plane A".

Page 4-39: Second line from the bottom, last term should be $-2.5 \leq x \leq 0$.

Page 4-53: Second line from the bottom, the word abscisses should be replaced by "arguments".

Page 4-58: Fourteenth line, first word should be "be".

Page 4-59: Third paragraph, first sentence should be "many finite functions do not have a maximum value".

Page 5-7: Figure 5-2-13, inserting follows:
$P_1(1,2)$, $P_2(3,6)$, $P_3(??)$, $P_4(7,4)$, $P_5(8,6)$, $P_6(??)$.

Page 5-36: Problem 7, example 1, second term, numerator should be "CE-BF".

Page 5-40: Problem 14-f, first part should be $3x + \sqrt{y} = 9$.

Page 5-55: Tenth line, adding "page 5-49:".

Page 5-58: Ninth line, delete first two words, adding "study carefully the following" . . .

Page 6-11: Insert footnote: * n!, read n factorial, is defined to be the product of 1x2x3 . . . x n. Example: 5! = 1x2x3x4x5 = 120.

Page 6-28: Problem 10, last term: $\sec^2 s + \csc^2 s$.

Page 6-34: Insert "1" on 12th line; delete Exercise 5.73; insert "2" on the 6th line from the bottom.

Page 6-35: Problem 6, $a \neq (2k-1)\pi$; Problem 7, $a \neq k\pi$.

Page 7-27: The beginning of the last paragraph, adding "Except problem 6,".

Page 7-41: The line above the solution should be $0 \varepsilon R \leftrightarrow (0,0) \varepsilon C$ the . . . The second line from the bottom should be $0^2 - b^2 + 1 \rightarrow b^2 = 1$.

Page 8-17: Problem 1 should be, $2x^3 - 3x^2 + 6x - 2 \div x - 4$.

Page 9-5: Footnote: A partial sum is the sum of a set of consecutive terms of a sequence.

Page 9-7: 1 (b) the upper limit of the summation should be 6. The third line from the bottom, adding a word "sigma" at the end of the line.

Page 9-10: Fourth line, last term should be $\left(\frac{1}{2}\right)^k$.

Page 9-17: Problem 5, the lower and the upper limits of the summation should be 1 and 5, respectively.

Page 9-18: Exercise 9-5-3, problem 3, last term should be 0.027.

Page 9-27: Exercise 9-6-3, delete the numbers 2 and 3 on line 8.

Page 9-30: First line, the coefficients $\frac{1}{16}$ should be squared.

Page 10-9: Figure 10-3-3, last upper number should be 5, not 81.

Page 10-12: Fifth line, at the end, adding $y \neq 0$.

Page 10-23: Seventh line from the bottom, the original symbols, $x \varepsilon 0$ should be $x \varepsilon R$.

Page 10-28: Problem 2a should be $(10)^{2x - 3} = 100$.

Page 10-29: Problem 4b add x at the end.

Chapter 1

Number Sets and Computer Programming

1-1 Introduction

Along with this text you will use an electronic digital computer as a tool for solving problems, developing insights, and making discoveries that otherwise might be quite difficult. You will use the computer as a device to effectively aid your learning. Such an approach should produce insight into the power of the computer as well as increase your understanding of algebra and trigonometry.

A digital computer is an organized collection of electronic components. It is capable of interpreting and executing a variety of minutely detailed instructions for manipulating numbers. The art of organizing sequences of these instructions to solve problems is called programming. You will communicate with the computer in an instruction language called BASIC, (Beginners All-Purpose Symbolic Instruction Code). The use of this programming language will force you into the habit of thinking and communicating precisely, leaving nothing to chance or misinterpretation. As you proceed you will find that programming is not merely concerned with writing computer instructions in BASIC. First must come an analysis of the step by step process for solving the problem at hand. This process may then be implemented as a program which the computer can execute.

Step by step problem solving procedures are known as algorithms, a word derived from the name of al-Khowarismi, a famous 9th century Asian mathematician who invented the "shift and add" algorithm for multiplying decimal numbers. Unfamiliar though the word may be to most of us, we all use algorithms in our daily lives. For example, an algorithm designed to get you to class each morning might be: get up, wash, dress, eat breakfast, run to the street corner, catch the school bus. Is each step absolutely essential? Not really; you could omit washing and breakfast. Clearly the algorithms of our daily living can be broad, flexible, and fuzzy. For computer problem-solving however, algorithms must be expressed as tightly organized, unambigous sequences of instructions which lead to answers in a finite amount of time. To help make a transition from the imprecise nature of a problem stated in English to the precise computer algorithm, programmers commonly use an intermediate step called flow charting. We will illustrate this procedure with an example drawn from everyday life.

1-2 Flow Charting

Suppose we are given the problem of describing, to another person, the procedure for sharpening a pencil. In order to clearly describe the sequence of steps to be carried out we will construct a flow chart. The beginning and end of the algorithm will be represented by a small oval. Each specific operation to be carried out will be contained in a rectangle. Our first flow chart of the algorithm is shown below.

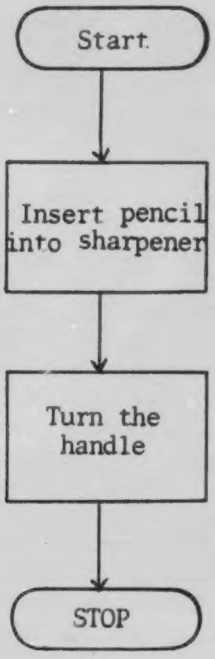


Figure 1-2-1

Careful examination of this flow chart reveals several inadequacies in the algorithm. For example, a person using this procedure could grind his pencil to a stub, as no indication is made concerning the number of times to turn the handle. A refined or more precise form for this algorithm may be seen in Figure 1-2-2.

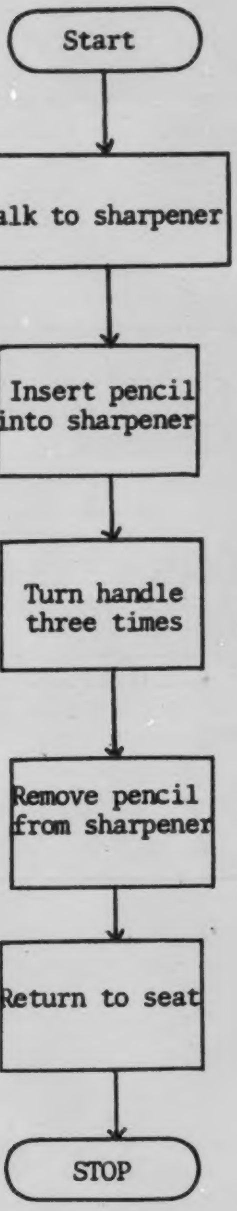

Start
Walk to sharpener
Insert pencil into sharpener
Turn handle three times
Remove pencil from sharpener
Return to seat
STOP

Figure 1-2-2

Anyone who has ever sharpened a pencil should recognize that turning the handle three times does not guarantee a sharp pencil. The procedure is more accurately described as one of turning the handle three times, removing the pencil, and answering the following question. "Is the pencil sharp?" If the answer is "no" the pencil is returned to the sharpener and the whole process is repeated. This repetition continues until we are able to answer the question with "Yes, the pencil is sharp". Only then do we return to our seat. Since our goal is to precisely describe the process of sharpening a pencil, we must introduce these ideas of decision making and repetitive operations into our flow chart. The diamond shaped figure is normally used in flow charting to represent the decision making step in an algorithm. This symbol is displayed in the refined flow chart in Figure 1-2-3.

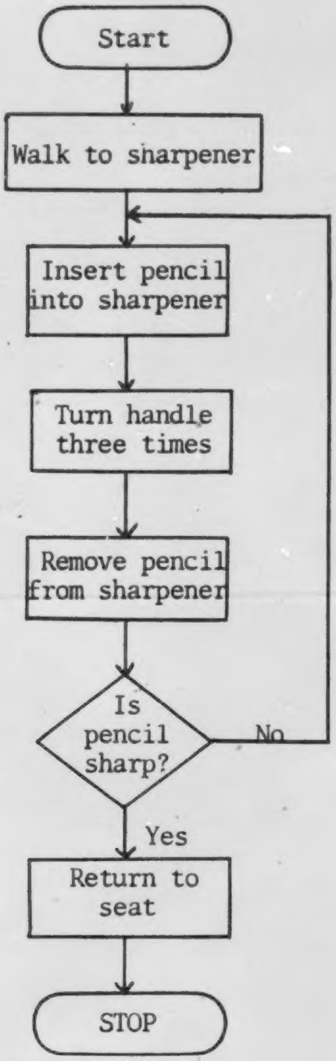


Figure 1-2-3

The flow chart illustrates how decision making may be used to direct repetitive operations. This procedure is called looping. A fourth refinement of this flow chart, shown in Figure 1-2-4, displays additional logic loops related to this algorithm.

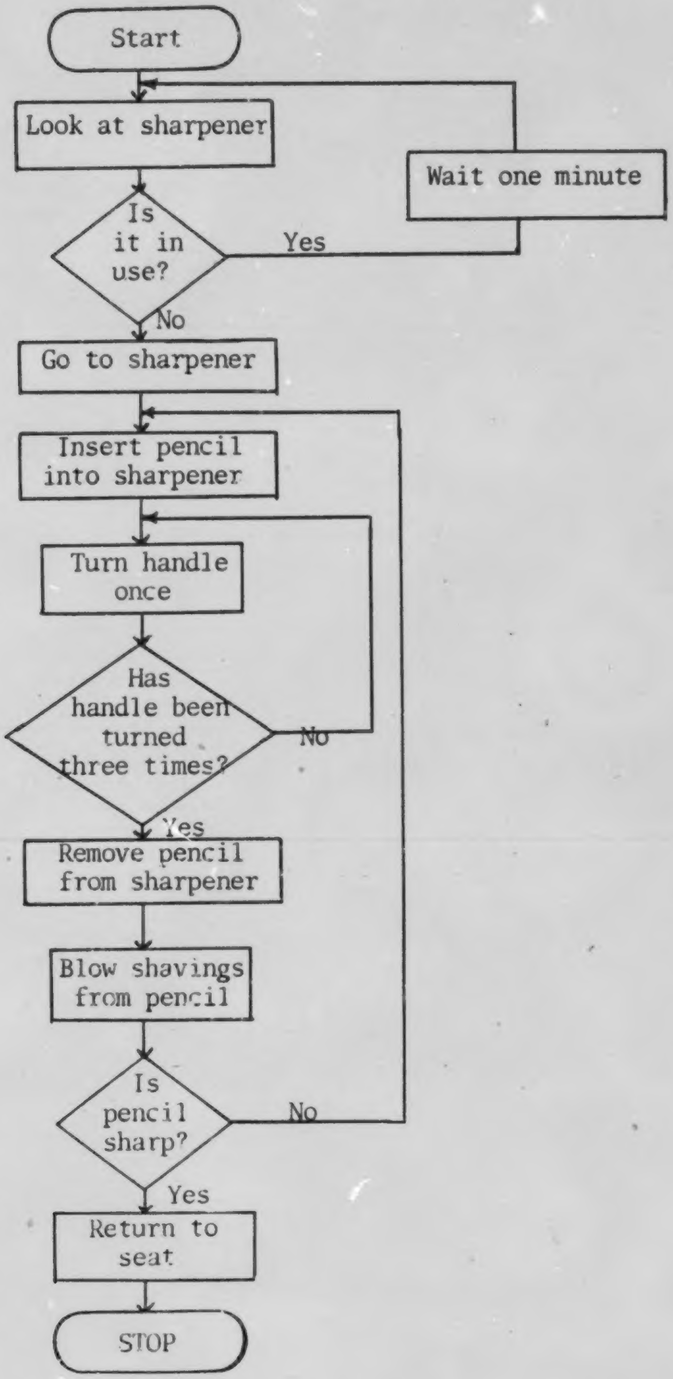


Figure 1-2-4

Exercise 1-2-5

Consider the flow chart shown below and answer the following questions.

1. Will this algorithm result in a sharpened pencil?
2. What is the essential difference between this flow chart and the one shown in Figure 1-2-4?
3. Which algorithm would you rather use? Why?

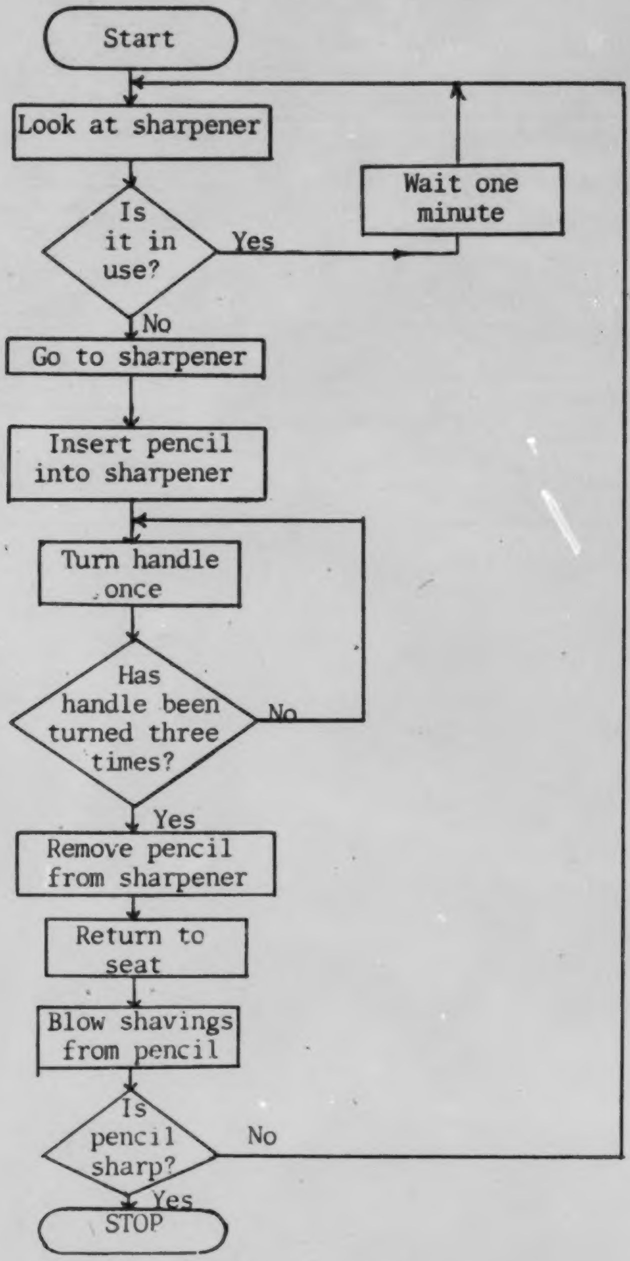


Figure 1-2-6

You should practice flow charting some situations in your daily life to get the experience of starting with a very loose description of a process and making successively more precise and specific flow charts. This cyclical process of refinement, of selecting and localizing significant detail, is essential to good flow charting technique. The more thorough the flow chart, the easier it is to translate the final flow chart into computer instructions.

Proble[illegible]et 1-2-7

Draw a preliminary (macro) flow chart, an intermediate flow chart, and a final (micro) flow chart for the following processes. Each flow chart should show a refinement of the logic in the preceeding flow chart.

1. The process of asking a girl for a date.

2. The process of opening a door.

3. The process of changing a tire.

4. The process of frying an egg.

5. The process of flow charting.

6 The process of choosing a marriage partner.

7. The process of dropping this course.

1-3 Sets

In algebra as in other branches of mathematics the concept of a set and the properties of sets are useful. "Set", an undefined term, means a collection of objects. The only requirement in using sets is that a set be described precisely. We must be able to determine whether or not a given object is a member of a given set. An object in a set is called a member or element of the set. The symbol, ε, is used to denote set membership.

Example 1-3-1

Given the set $A = \{4, 1, 7, 9, -13, -8\}$
$7 \varepsilon A$ means 7 is an element of the set A; while,
$0 \not\varepsilon A$ means 0 is not an element of the set A.

True or false?

1. $9 \varepsilon A$
2. $-13 \not\varepsilon A$
3. $-5 \not\varepsilon A$
4. $x \varepsilon A$ (where x is an integer)
5. $x \not\varepsilon A$ (where x is an integer)

You should have had no trouble in judging statements 1-3 above as true or false. However, sentences 4 and 5 may have caused you some concern. In fact, it is impossible to judge them as true or false until some substitution of a number name is made for x. You may recall that such sentences are sometimes referred to as "open sentences".

> A set may be specified by listing the names of all its members in braces, {}, as in Exercise 1-3-1 or, as will be seen later, by stating a rule for selecting its members.

Frequently we will have need to refer to all of the elements of a given set A. In doing so, we will find ourselves making repetitive statements such as; "for each $x \in A$", "for all $x \in A$", or, "for every $x \in A$". In order to simplify the handling of such statements we will replace them all with the notation $\forall\, x \in A$.

Example 1-3-2

$\forall\, x \in A$ means "for each x,an element of the set A".

One of the concepts associated with sets is that of a subset. If all of the members of a set A are also members of a set B, then set A is said to be a subset of set B, denoted by $A \subseteq B$.

Definition 1-3-3 Subset

Given two sets A and B, $A \subseteq B$, if and only if $\forall\, x \in A$, $x \in B$.

Example 1-3-4

Given $A = \{+1,+2,+3\}$
$B = \{-1,+1,-2,+2,-3,+3\}$

Is $A \subseteq B$?

We can see that $1 \in A$ and $1 \in B$
and $2 \in A$ and $2 \in B$
and $3 \in A$ and $3 \in B$
and there are no other elements in A.
$\therefore \forall\, x \in A$, $x \in B$ so $A \subseteq B$.

The set having no members is called the empty set or null set, written $\emptyset$. The null set is a subset of every set.

Example 1-3-4

Given two finite, non-empty sets, A and B, flow chart the process for determining if $A \subseteq B$.

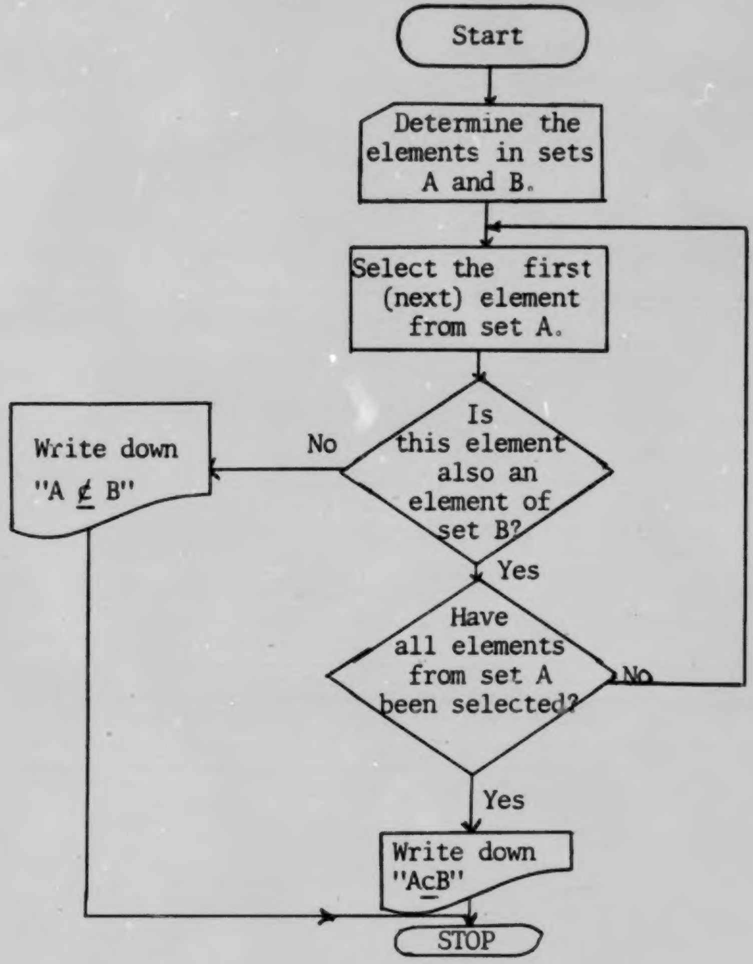

Start
Determine the elements in sets A and B.
Select the first (next) element from set A.
Is this element also an element of set B?
No
Yes
Write down "$A \nsubseteq B$"
Have all elements from set A been selected?
No
Yes
Write down "$A \subseteq B$"
STOP

Figure 1-3-5

Two new symbols are illustrated in the previous flow chart. The first one, a rectangle with the upper lefthand corner cut off was used to indicate that the sets A and B must be specified before we can determine if $A \subseteq B$. A box of this general shape will always be used whenever information is required as an integral part of an algorithm. It will be referred to as an input box. The second new shape, a rectanglar shape with a curved base called a print out box, is used whenever the results of an algorithm are to be written down. In the previous example, this symbol was used twice. Once to record the fact that $A \subseteq B$ and once to record the fact $A \nsubseteq B$ which ever conclusion was applicable.

Exercise 1-3-6

Follow the algorithm shown in Figure 1-3-5 and determine if $A \subseteq B$ for each of the following cases. Don't make your decision on the basis of casual observation of the sets. Go through the complete algorithm for each example. If you don't, you may have difficulty with the problems to come.

1. A = {0,Ø,$,#,1,Q,K}
   B = {0,Ø,$,K,Q,1,#}

2. A = {1,9,7,$,2}
   B = {$,7,3}

3. A = {#,$,@,←,/}
   B = {/,$,←}

4. A = {/,$,←}
   B = {#,$,@,←,/}

5. How must the flow chart in Figure 1-3-5 be modified when the restriction that sets A and B are non-empty is removed?

Two important concepts with which you are probably already familiar are the union and intersection of sets. The union of two sets A and B is the set of all elements contained in either A or B. It is denoted by $A \cup B$. The intersection of two sets A and B is the set of all elements contained in both A and B. It is denoted by $A \cap B$. The formal definitions are given below.

Definition 1-3-7 Union of Sets

Given two sets A and B

$x \varepsilon (A \cup B)$
if and only if
$x \varepsilon A$ or $x \varepsilon B$

Definition 1-3-8 Intersection of Sets

Given sets A and B

$x \varepsilon (A \cap B)$
if and only if
$x \varepsilon A$ and $x \varepsilon B$

Example 1-3-9

Given $A = \{\uparrow, \$, \#, \leftarrow, @, ?\}$
$B = \{T, \leftarrow, /, !, @\}$

List the elements in $A \cap B$ and $A \cup B$.
$A \cap B = \{\leftarrow, @\}$
Each element is a member of set A and a member of set B.
$A \cup B = \{\uparrow, \$, \#, \leftarrow, @, ?, T, /, !\}$
Each element is a member of set A or a member of Set B.

Example 1-3-10

Construct a flow chart for the process of forming the union of two finite sets A and B. Be sure to illustrate all inputs, printouts, decisions and operations in the algorithm.

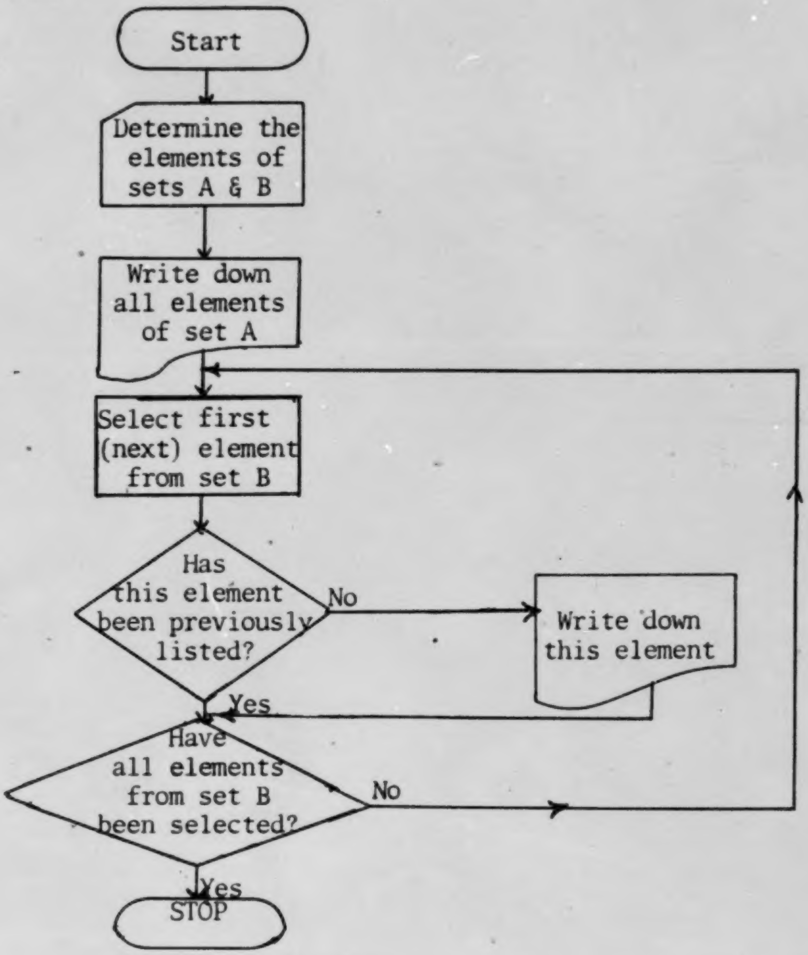

Start
Determine the elements of sets A & B
Write down all elements of set A
Select first (next) element from set B
Has this element been previously listed?
No
Write down this element
Yes
Have all elements from set B been selected?
No
Yes
STOP

Figure 1-3-11

Exercise 1-3-12

Follow the algorithm shown in Figure 1-3-11 and form $A \cup B$ for each of the following cases. Don't work this problem by casually observing the sets. Go through the complete algorithm for each example.

1. $A = \{?, \$, !, \#, \leftarrow\}$
   $B = \{T, \$, \#, 1\}$

2. $A = \{?, T, \$\}$
   $B = \{@, \leftarrow, \uparrow, \emptyset\}$

3. $A = \{1, 2, T\}$
   $B = \{1, T, 2\}$

Exercise 1-3-13

1. Construct a flow chart for the process of forming the intersection of two finite sets A and B. Be sure to illustrate all inputs, printouts, decisions, and operations in the algorithm.

2. Follow your algorithm for forming the intersection of sets, $A \cap B$, and test its validity by using the following sets.

   a. $A = \{?, \$, !, \#, \leftarrow\}$
      $B = \{X, T, \$, \#, 1\}$

   b. $A = \{?, \#, A, \leftarrow, B\}$
      $B = \{T, !, \$, @\}$

Problem Set 1-3-14

Given $A = \{0, 2, 4, 6, 8\}$
$B = \{0, 1, 2, 3, 5, 7, 8\}$
$C = \{4, 6, 9\}$

List the elements in each set.

1. $A \cap B$
2. $A \cup B$
3. $B \cap C$
4. $B \cup C$
5. $(A \cap B) \cup C$
6. $(A \cup B) \cup C$
7. $(B \cap C) \cap A$
8. $(B \cup C) \cap A$
9. $A \cap \emptyset$
10. $A \cup \emptyset$
11. Form all possible subsets of set A.
12. Is it true that $A \subseteq (B \cup C)$?
13. Is it true that $B = A \cup C$?
14. Is it true that $B \subseteq A \cap C$?
15. Is it true that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$?

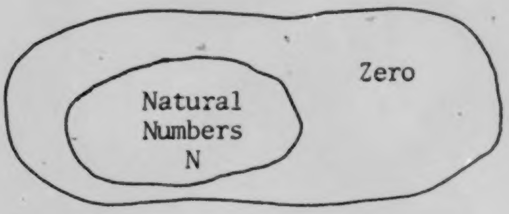
16. Draw a flow chart of the algorithm for forming each of the following sets.
    a) $A \cap (B \cup C)$
    b) $(A \cup B) \cap (A \cup C)$
17. Draw a flow chart of the algorithm for determining if $A \subseteq (B \cup C)$.
18. In a class of 23 students, 15 take Geometry, 10 take Algebra, and 12 take Latin. There are 6 students that take Geometry and Latin, 7 that take Latin and Algebra, and 5 that take Algebra and Geometry. There are 2 students that take all three courses.
    a) How many students take exactly 2 of these courses?
    b) How many students do not take any of these courses?

1-4 The Natural Numbers

No matter how or where you studied a first course in algebra the number system was gradually extended from the set of natural numbers, 1, 2, 3,..., through the sets of whole numbers, integers, rational numbers, and finally to the set of real numbers. In this section we will briefly review natural numbers and learn to write simple computer programs.

The names used for counting numbers ('one', 'two', etc.) should, of course, be familiar to us all. We should also be familiar with the symbols, called numerals ('1', '2', etc.). In addition, we should know that numerals such as '1/2', '3/4', '-3', '-2/3', 'π', '$\sqrt{2}$', and '4^3' also represent numbers. Different symbols may represent the same number. For example, '2', '4/2', '6/3', 'II', '$\sqrt{4}$', '1 + 1', and '8 - 6' are all names for the same number. A number is an idea which may have many names.

The set of numbers {0,1,2,...} is called the set of whole numbers. The subset {1,2,...} is called the set of natural numbers, as illustrated in Figure 1-4-1. Since we will be making frequent reference to the set of natural numbers it will be convenient to let the letter N stand for the set of all natural numbers. Thus, N = {1,2,3,...}.



The Set of Whole Numbers

Figure 1-4-1

This set of natural numbers, N, is precisely the set of numbers obtained by starting with the number one and repeatedly adding one. This process for generating the set of natural numbers is based upon an important property, called the successor property. That is, given any natural number it is always possible to generate the next natural number by adding one. This new number is called the successor of the original number.

Example 1-4-2

Given any natural number, x, flow chart the algorithm for generating its successor.

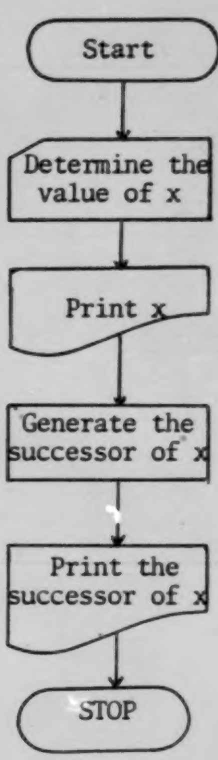


Figure 1-4-3

We will now introduce the BASIC computer language and show how the algorithm, from Figure 1-4-3, can be communicated to the computer. The first symbol, following Start, is an input box which indicates that information must be input into the computer. That is, we must supply the number whose successor is to be generated. This is done, in BASIC, through the combination of two statements, READ and DATA.

The first statement, READ, is one which defines a variable to be used within the algorithm of the program. The second statement, DATA, assigns a numerical value to this variable.

```
READ X

DATA 237
```

When the computer encounters the two instructions shown above, the number 237 is substituted into the variable X. These two statements, READ and DATA, are always used in conjunction with one another. They replace the input symbol of a flow chart.

The next symbol we encounter in our algorithm is the output box. It is used to represent those points in an algorithm at which the computer is to print, or output, information. It is replaced by the simple instruction, PRINT X. When the computer encounters this instruction, it will print the numerical value of X, in this case 237. Our BASIC program to this point is:

```
READ X

DATA 237

PRINT X
```

The rectangular box in our flow chart, which reads "Generate the successor of X" indicates that an operation must be carried on internally within the computer. We command the computer to carry out this operation by the following instruction,

```
LET S = X + 1
```

This instruction is interpreted by the machine to mean, take the value of X, add one to it, and substitute this number into the variable S. Our BASIC program has now become:

```
READ X

DATA 237

PRINT X

LET S = X + 1

PRINT S
```

Notice that PRINT S has been added to the program. This instruction represents the last output symbol in our flow chart. It directs the computer to print the successor of 237, in this case, 238.

The BASIC program of our algorithm will be complete after making two minor additions. The first is the numbering of the instructions. This indicates to the computer the sequence in which the instructions are to be carried out. The second is the addition of the last line of the program. This is the instruction END. It indicates to the computer that there are no more instructions to be interpreted. END must always be the last instruction in any BASIC program.

Our complete program for this algorithm is shown below, along with the printout from an actual computer run.

```
10  READ X

20  DATA 237

30  PRINT X

40  LET S = X + 1

50  PRINT S

60  END


RUN

SESSOR            12:30      D2  Thu  07/18/68

237

238

RAN  0.1 SEC.

READY
```

We will now write a variation of the program for this algorithm. This will be for the purpose of discussing additional aspects of these same instructions.

```
11111  READ A

11116  DATA 237

11121  PRINT A;

11126  LET A = A + 1  (refer to page 1-18)

11131  PRINT A

11136  END
```

Notice the difference in line numbers between the two programs. Any numbering of lines can be used as long as they are whole numbers less than 100,000. They only exhibit the order of execution of the instructions. Also notice that any of the 26 letters of the alphabet may be used as the variable taking on the numerical value of the number whose successor is to be computed. In this case we use A instead of X.

The first print statement is followed by a semicolon. This will cause the second PRINT to be executed on the same line as the first. That is:

```
237         238
```

One may also choose to use a comma after a print statement. This will accomplish the same objective as the semicolon, only there will be more space between the number and its successor.

```
PRINT X

LET S = X + 1

PRINT S
```

produces output of

```
237

238
```

while

```
PRINT X;

LET S = X + 1

PRINT S
```

produces output of

```
237   238
```

and

```
PRINT X,

LET S = X + 1

PRINT S
```

produces output of

```
237                 238
```

Let us now discuss line 1126 in the second program, LET A = A + 1. The LET instruction is always used to establish a new value for a variable. It must be written in the form:

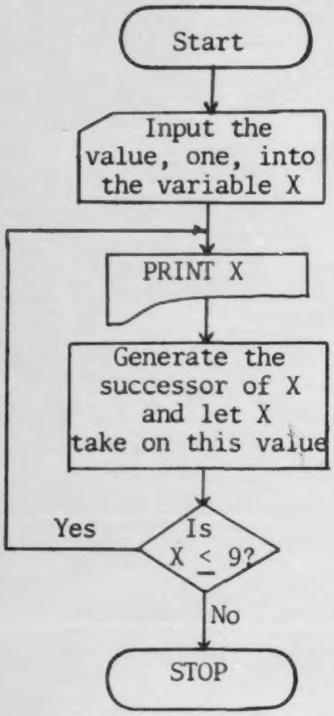
LET (single variable) = (algebraic expression).

The expression, on the right side of the equality is evaluated and this new value is substituted into the variable on the left replacing the original value. This technique is used to change the value of the variable within one instruction. Thus, we can eliminate the need for two variables which existed in the first program. The instruction, LET A = A + 1, causes the value of A to be replaced by the value of its successor.

Now that we have learned some simple programming, we will use the computer to produce subsets of the set of natural numbers, N.

Example 1-4-4

Flow chart the algorithm for producing the first nine natural numbers by generating successors and transcribe this algorithm into a BASIC program.


Start
Input the value, one, into the variable X
PRINT X
Generate the successor of X and let X take on this value
Yes
Is X ≤ 9?
No
STOP

```
10  READ X

20  DATA 1

30  PRINT X;

40  LET X = X + 1

50  IF X < = 9 THEN 30

60  END
```

Figure 1-4-5

Notice the use of a new BASIC instruction, IF -- THEN ---, which corresponds to the decision box in the flow chart. This instruction must always take the form;

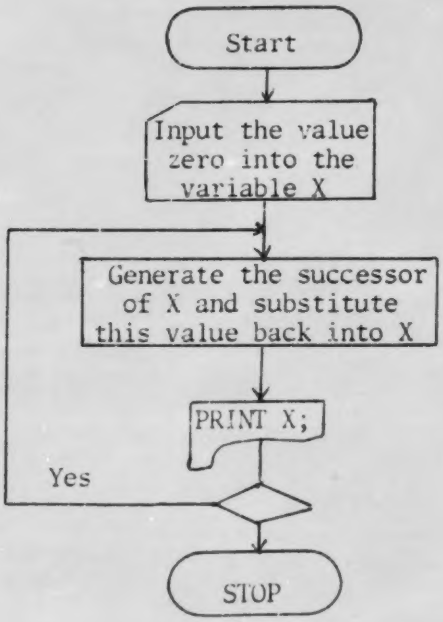
IF (algebraic expression) (relation) (algebraic expression) THEN (line number)

Whenever the sentence following the word IF is true the computer will jump to the instruction whose line number follows the word THEN. When the sentence following the word IF is false, the computer will continue on to the next instruction. In this example, wherever the statement X < = 9 is true the computer will jump to program line 30. However, when X < = 9 is a false statement the computer will move on to line 60. The sentence X < = 9 is read "X is less than or equal to 9." The BASIC notation for the standard inequality statements from algebra is shown below

<	less than
>	greater than
< =	less than or equal to
> =	greater than or equal to
< >	not equal to

Exercise 1-4-6

In Figure 1-4-7 you will see a modified flow chart of the algorithm, described in Example 1-4-4, for producing the first nine natural numbers by generating successors. Write the modified BASIC program for this new flow chart after completing the missing statement in the decision box. After the program has been written, answer the question below it.


Start
Input the value zero into the variable X
Generate the successor of X and substitute this value back into X
PRINT X;
Yes
STOP

PROGRAM

Figure 1-4-7

Exercise 1-4-6 (continued)

What are the differences between the IF-THEN instructions in the two programs, Example 1-4-4 and Exercise 1-4-6?

Before beginning our programming of some concepts relating to the natural numbers let's consider two interesting subsets of the natural numbers;

$$A = \{X | X = 2n, n \in N\}$$

and

$$B = \{X | X = 2n - 1, n \in N\}$$

The notation for the above sets is called set builder notation. The first case is read "Set A is the set of all elements X such that X is product of a natural number and two." That is, A is the set of even natural numbers; B is the set of odd natural numbers.

Exercise 1-4-8

1. Construct a flow chart algorithm for the following problems and write the corresponding BASIC program.
    a. Generate the first 17 odd natural numbers.
    b. Generate the first nine even natural numbers and find their sum. Remember that zero is not a natural number.
2. Perform the operations indicated in the following algorithm before proceeding any further in the text.

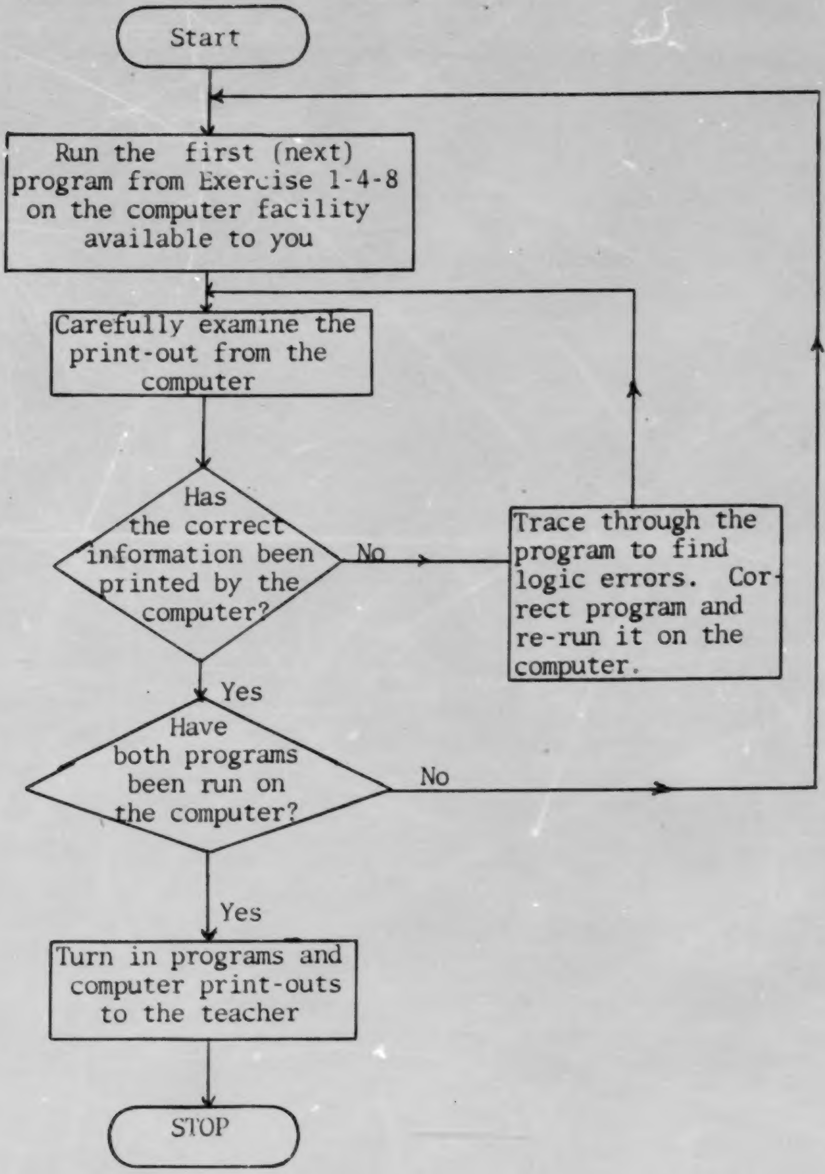

Start
Run the first (next) program from Exercise 1-4-8 on the computer facility available to you
Carefully examine the print-out from the computer
Has the correct information been printed by the computer?
No
Trace through the program to find logic errors. Correct program and re-run it on the computer.
Yes
Have both programs been run on the computer?
No
Yes
Turn in programs and computer print-outs to the teacher
STOP

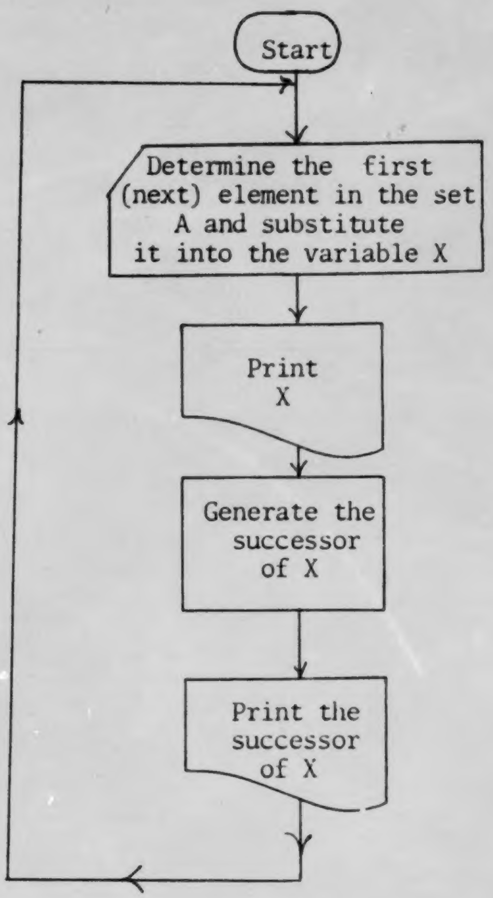
Figure 1-4-9

In order to introduce one more BASIC instruction we turn our attention back to the topic of natural number successors. Suppose you are asked to find the successor for each element in a finite subset of the natural numbers.

Example 1-4-10

Given A = {3,27,99,132,1974,94368}

Generate the successor for each element in set A.



Start
Determine the first (next) element in the set A and substitute it into the variable X
Print X
Generate the successor of X
Print the successor of X

```
 5   READ X
10   DATA 3,27,99,132,1974,94368
15   PRINT X;
20   LET X = X + 1
25   PRINT X
30   GØ TØ 5
35   END
```

Figure 1-4-11

The complete output of the program from Figure 1-4-11 may be seen below

```
SUCCES           10:43    D17    Thu    11/18/68

     3       4
    27      28
    99     100
   132     133
  1974    1975
 94368   94369
 OUT OF DATA LINE 5

 RAN    0.1 SEC.
 READY
```

Figure 1-4-12

Examination of the program in Figure 1-4-11 reveals a new BASIC instruction.

```
30 GØ TØ 5
```

This is an unconditional jump instruction which directs the computer back to line 5. This means the READ X instruction will be encountered repeatedly by the computer. The first time, the number 3 is substituted into X. The second time, the number 27 is substituted into X, and so on until the sixth time when 94368 is substituted into X. The seventh time the READ X instruction is encountered, there are no unused data items left in line 10. The computer prints "OUT OF DATA LINE 5" and jumps to line 35. This is a valid procedure for terminating a computer program.

The BASIC language uses operation symbols which parallel those we are familiar with from algebra. The only obvious differences are the multiplication and exponentiation operation symbols. In this programming language the asterisk, *, is used as a multiplication symbol and the "up arrow", ↑, indicates exponentiation.

Example 1-4-13

Operation	Symbol	Algebraic Expression	BASIC Expression
Addition	+	A + B	A + B
Subtraction	-	X - Y	X - Y
Multiplication	*	MN	M * N
Division	/	(A - B)/C	(A + B)/C
Exponentiation	↑	$(R + S)^T$	(R + S)↑T

Exercise 1-4-14

1. Which of the following are not legal expressions in BASIC? Explain why.

   a. A + (3/C) * D
   b. 3 * (A/+8Y)
   c. $4^2 * 3 + 2(1 - 4)$
   d. ((R + 2 ÷ 3) ↑2) - XY
   e. (Q + I - 4)/X + Y↑4

   Write a BASIC expression that is equivalent to the mathematical expressions given in Problems 2, 3, and 4.

2. $\frac{(4x3) - (5x7)}{6 + 2}$

3. $(4+3) \div 2^3$

4. $(2+2+3)^{2+1}/(3-1)^{1+1}$

   Write a mathematical expression that is equivalent to the BASIC expressions given in problems 5 and 6.

5. 1/(1+2/(X+3/(X+4/X)))

6. (2+3)↑2+3

Problem Set 1-4-15 (read page 1-28)

Write BASIC programs for each of the following problems.

1. Print the first five natural numbers.
2. Print the first seven odd natural numbers.
3. Print the first 11 even natural numbers and find their sum.
4. Print the first 9 odd natural numbers and find their sum.
5. Compute the sum of the first 2, 3, 4, 5, 6, 10, 20, and 510. odd natural numbers. Analyze the answers and write an expression for the sum of the first n odd natural numbers.
6. Compute the sum of the first 2, 3, 4, 5, 6, 10, 20, and 510 even natural numbers. Analyze the answers and write an expression for the sum of the first n even natural numbers.

*7. Write a program to sort the following set of natural numbers into odd numbers and even numbers. {5,3,6,4,16,21,1006,14}

*8. a. Write a program that prints the first N terms of the Fibonacci sequence. Input the value for N into the program with a READ-DATA statement. The Fibonacci sequence, 1, 1, 2, 3, 5, 8, ..., has the property that each number (after the first two) is the sum of the two previous numbers.
   b. Modify your program in (a.) above to also compute and print out the sum of the first N terms.

Problem Set 1-4-15 (continued)

c. Write a program which will take this Fibonacci sequence and find the first 20 ratios of a term to the term preceeding it. Then, calculate and print out the value for the expression $(2R - 1)^2$ for each of the 20 Ratios, R, found.

* Stars indicate difficult problems.

1-5 Integers

As long as man's only need for numbers was to keep track of his possessions, the counting numbers were adequate. When his neighbor wanted to barter and he lacked one object to complete the bargain, he needed a new number. This new number had to represent owing one object instead of owning it. Perhaps this is the manner in which the "opposite" of a natural number was born. This opposite of a natural number is called a negative number. If the neighbor bartered for all of a man's possessions and none were left, a need for zero arose. All of these numbers are called integers. The set of integers, I, is the union of the set of natural numbers, the set of all opposites of the natural numbers and the set containing zero.

Definition 1-5-1 The Set of Integers

The set of integers I is defined as

$$I = \{x|x \varepsilon N\} \cup \{-x|x \varepsilon N\} \cup \{0\}$$

In the previous section, the concept of the successor of a natural number was developed. We know that this concept also applies to the set of integers. That is, given any $x \varepsilon I$, the successor may be found by adding the number one. In Example 1-4-4 we programmed an algorithm for producing the successive natural numbers from one to nine. This is a valuable technique in programming, one which is associated with the counting process. This same technique could also be used to direct the computer to produce successive elements from the set of integers. However, there is a simple instruction in BASIC which will accomplish this same task. This instruction is called the FOR-NEXT loop. It is used to generate sequences of numbers by addition.

Any use of the FOR-NEXT instruction requires <u>two</u> program lines. It must be written in the following form.

```
(line number)  FOR (variable) = (expression) TO (expression) STEP (expression)
      .          .
      .          .
      .          .
(line number)  NEXT (same variable)
```

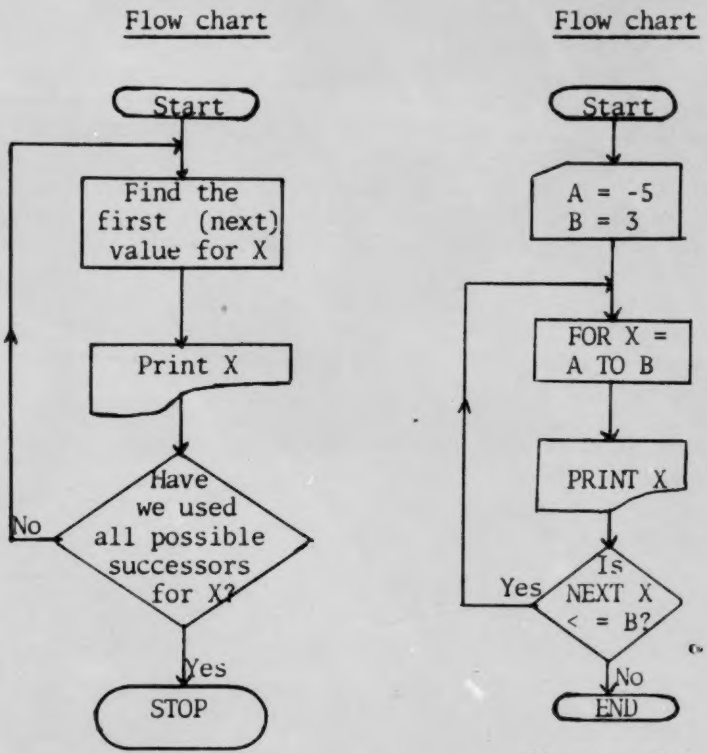
Example 1-5-2

```
40 FOR X = A TO B  STEP 1
 .
 .
 .
70 NEXT X
```

Line 40 instructs the computer to substitute the value of A into the variable X and proceed through the program. The computer then executes all instructions between line 40 and line 70. When the NEXT X instruction in line 70 is encountered, the computer analyzes the following question. "Is there another successive value for X which is less than or equal to B?" If the answer is yes, the successor of the value of X is substituted as a new value for X. The computer then loops back to line 40 and the process is repeated. This repetition is continued until the answer to the question is: "No, the next successive value for X is not less than or equal to the value of B." At this point the computer stops looping back and proceeds to execute the instruction following line 70.

Example 1-5-3

Write a BASIC program which will direct the computer to print the successive integers from -5 to +3 inclusive.



Flow chart

Flow chart

Program

```
20  READ A,B
30  DATA -5,3
40  FOR X = A TO B
50  PRINT X;
70  NEXT X
80  END
```

OUTPUT

```
-5,-4,-3,-2,-1,0,1,2,3
```

Figure 1-5-4

The READ A,B instruction in line 20 directs the computer to obtain values for the two variables A, and B from the DATA statement in line 30. Hence -5 is substituted into A and +3 is substituted into B.

The STEP 1 part of the instruction in line 40 has been omitted. When this is done, the computer automatically assumes a STEP of one. Any other increment may also be used, but it must be specified with the STEP notation. For example, if line 40 is changed to (40 FOR X = A TO B STEP 2), the print out becomes (-5, -3,-1,1,3). Negative steps may also be used for counting backward.

Exercise 1-5-5

1. a) What element in the program in Example 1-5-3 forces the computer to print all of the numbers on the same line?
   b) How must the program in Example 1-5-3 be changed so that the numbers would be printed in a single vertical column?

2. Work through each of the following programs by hand. Follow the instructions exactly the way the computer would. Produce the output from each program in exactly the same form as that which the computer would produce.

a)
```
20  READ A,B
30  DATA -5,3
40  FOR X = A TO B
50  LET Y = X+5
60  PRINT Y;
70  NEXT X
80  END
```

b)
```
20  READ A,B
30  DATA -5,3
40  FOR X = A+5 TO B+5
60  PRINT X;
70  NEXT X
80  END
```

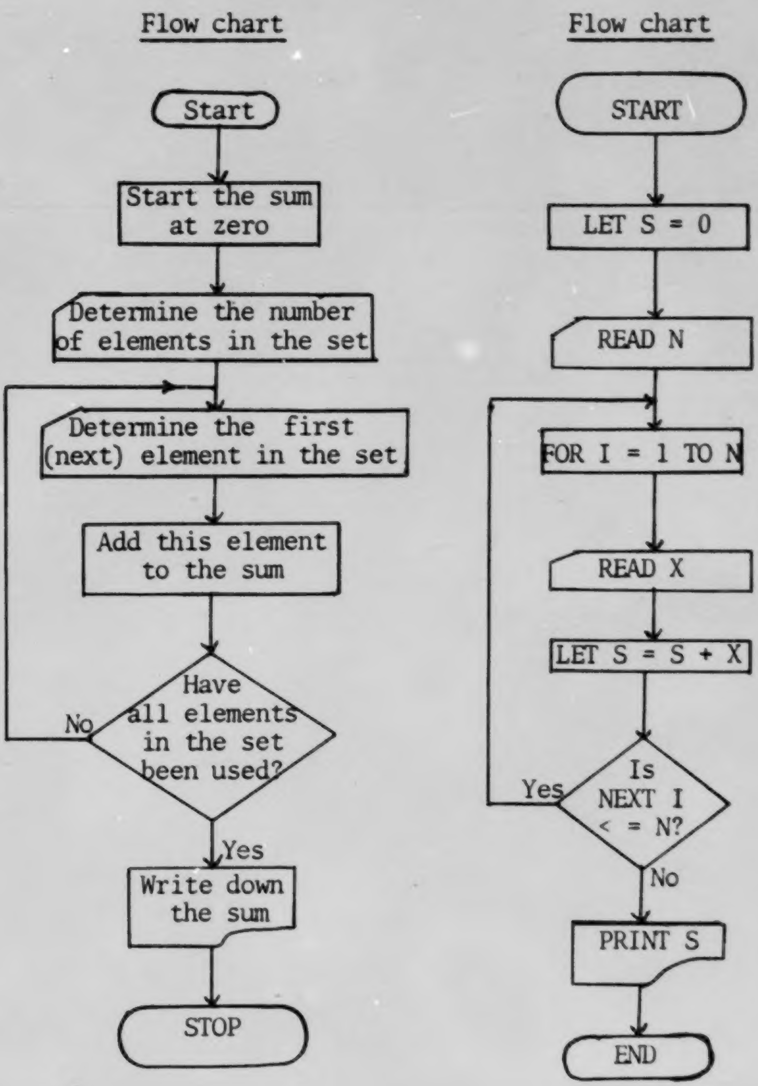
c)
```
 13  READ X,Y
 15  DATA 3,1
 21  FOR I = -X TO Y*4
 76  LET M = I*7
219  PRINT M
900  NEXT I
999  END
```

d)
```
400  READ X,Y
410  DATA -9,1
420  FOR I = -X TO Y STEP -3
430  PRINT I;
440  NEXT I
450  END
```

Your work with Exercise 1-5-3 should produce insight into the power and flexibility of the FOR - NEXT instruction. It can be used in programming whenever a finite number of repetitive loops are involved in the algorithm. An example of the use of the FOR - NEXT instruction might be in finding the sum of all the elements in a finite set of non-sequential integers.

Example 1-5-6

Given the set of integers A = {-4,2968,45,-12,89,98,12,2,3,4,5,-2959,26,-1}, write a BASIC program for finding the sum of all the elements in the set.



Program

```
10  LET S = 0
20  READ N
30  DATA 14
40  FOR I = 1 TO N
50  READ X
60  LET S = S + X
70  NEXT I
80  PRINT "THE SUM IS";S
90  DATA -4,2968,45,-12,89
92  DATA 98,12,2,3,4,5
94  DATA -2959,26,-1
99  END
```

OUTPUT

```
THE SUM IS 276.
```

Figure 1-5-7

There is an interesting function in algebra called the "bracket" or "integer" function. It is formed with the equation $Y = [X]$. This equation is always interpreted to mean:

Y is the largest integer which is less than or equal to X.

Example 1-5-8

Sample values for X	Associated values for $[X]$
2.3	2
19	19
7.6204	7
13.9821	13
-4.3	-5
-7.914	-8

This same "bracket" function exists in BASIC. The only difference is that we write INT(X) in place of $[X]$. Whenever the computer encounters the expression INT(X), it substitutes in the largest integer which is less than or equal to X.

Example 1-5-9

The following program shows how the INT(X) expression can be used.

```
Program                                   Output

10  FOR I = 1 TO 2
20  READ X
30  DATA 3.741, -4.3                      3.741      3      8
40  PRINT X; INT(X); INT(X)+5             -4.3      -5      0
50  NEXT I
60  END
```

The INT(X) expression is an extremely powerful element in BASIC. It is used to round off numbers and to check for divisibility of one number by another.

Example 1-5-10

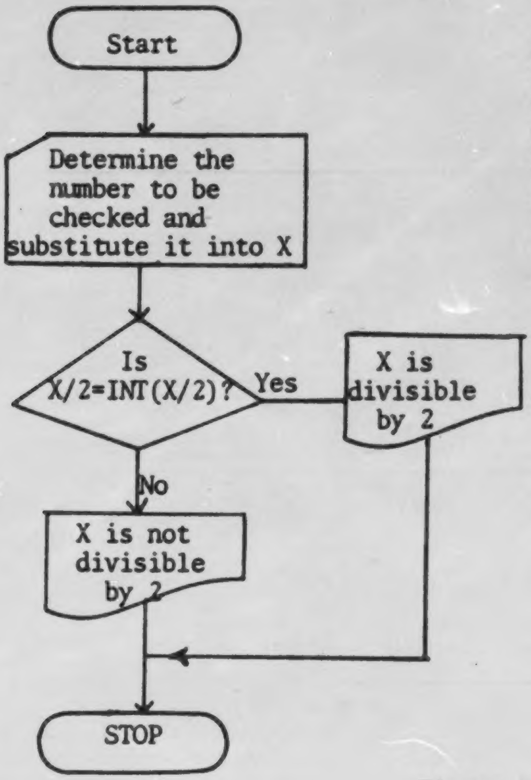
Write a BASIC program to determine if a given real number X is divisible by 2.

Whenever a number X is divisible by 2, the quotient will be an integer. However, if X is not divisible by 2, the quotient is not an integer. This test for divisibility can be expressed as follows using the INT(X) expression.

If X/2=INT(X/2) then X is divisible by 2.
If X/2≠INT(X/2) then X is not divisible by 2.

Hence, the following program will determine divisibility by 2.

Flow chart



Program

```
10  READ X
20  DATA
30  IF X/2=INT(X/2)  THEN 60
40  PRINT X;"NOT DIVISIBLE BY 2."
50  GO TO 70
60  PRINT X;"DIVISIBLE BY 2."
70  END
```

Figure 1-5-11

Exercise 1-5-12

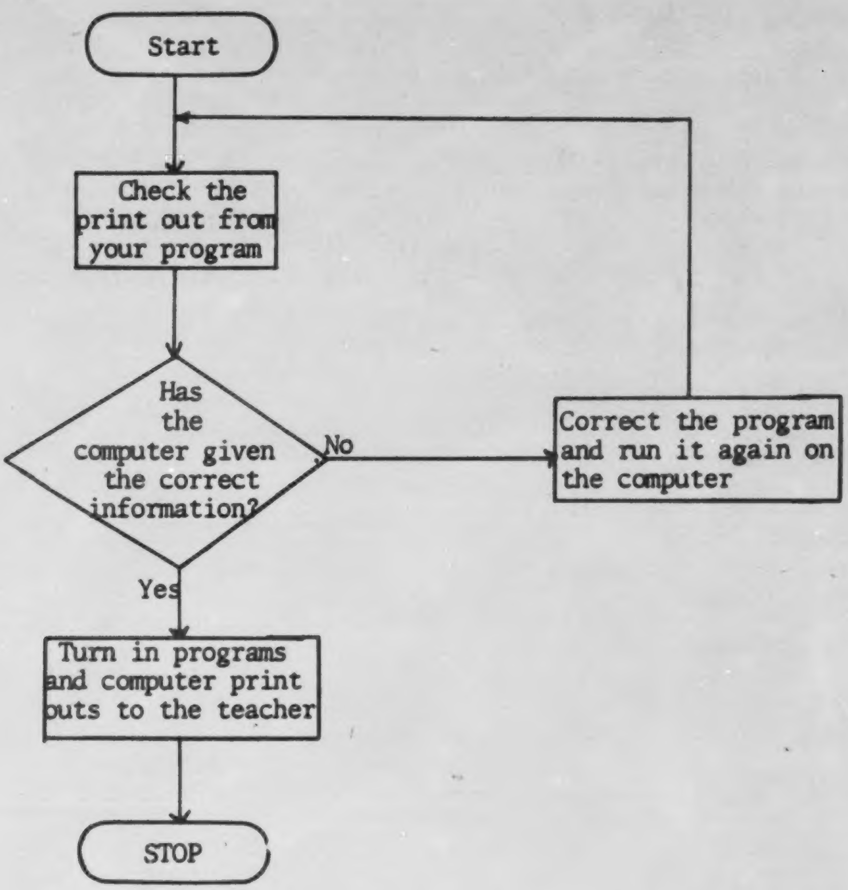
1. Write a BASIC program which will check all of the elements in a finite set of integers and print only those that are divisible by 13.

2. Run your program on the computer facility available to you using the following set as sample DATA.
   {221, 36, 39, -533, 27643, -65, 237, -314, 169, -225, 226}

3. Perform the operations in the following alogrithm before proceeding any further in the text.

Exercise 1-5-12 (continued)

3. (continued)


Start
Check the print out from your program
Has the computer given the correct information?
No
Correct the program and run it again on the computer
Yes
Turn in programs and computer print outs to the teacher
STOP

DO NOT GO ON UNTIL YOU HAVE WORKED THIS EXERCISE CORRECTLY BY YOURSELF.

Problem Set 1-5-13

1. Write a program to determine all of the integers n in the interval $A \leq n \leq B$, $A, B \in I$ such that $(n+1)/3 \in I$.

2. Write a program to produce the first nine multiples of 331.

3. Given any positive integer n, write a program to produce all positive integers which divide n.

4. Given any positive integer n, write a program which will determine if n is a prime number. A prime number is a positive integer, n, $n \neq 1$, which is divisible only by itself and 1.

Problem Set 1-5-13 (continued)

5. Any positive integer n may be written as a product of prime factors. That is, $n = A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdots A_r$ where A_1 through A_r are prime numbers. Write a program which will print the prime factors of any positive integer n.

We will now review the operations of addition and subtraction on the set of integers. You will recall from previous work in algebra that the absolute value of an integer, a, is a if $a \geq 0$. However, the absolute value of an integer, a, is -a if $a < 0$. For example, $|2| = 2$, and $|-3| = -(-3) = 3$. Hence we see that the operation of "absolute value" assigns a whole number to every integer.

Exercise 1-5-14

1. Prepare a flow chart to produce the absolute value of any integer.

2. Write a program which will evalute the expression $|x|$, where x is any integer.

Assuming that we know how to add and subtract whole numbers, we can define addition and subtraction on the set of integers.

Definition 1-5-15 Addition and Subtraction of Integers. $\forall a \forall b$, $a, b \in I$

1. If $a \geq 0$ and $b \geq 0$ then $a + b = |a| + |b|$

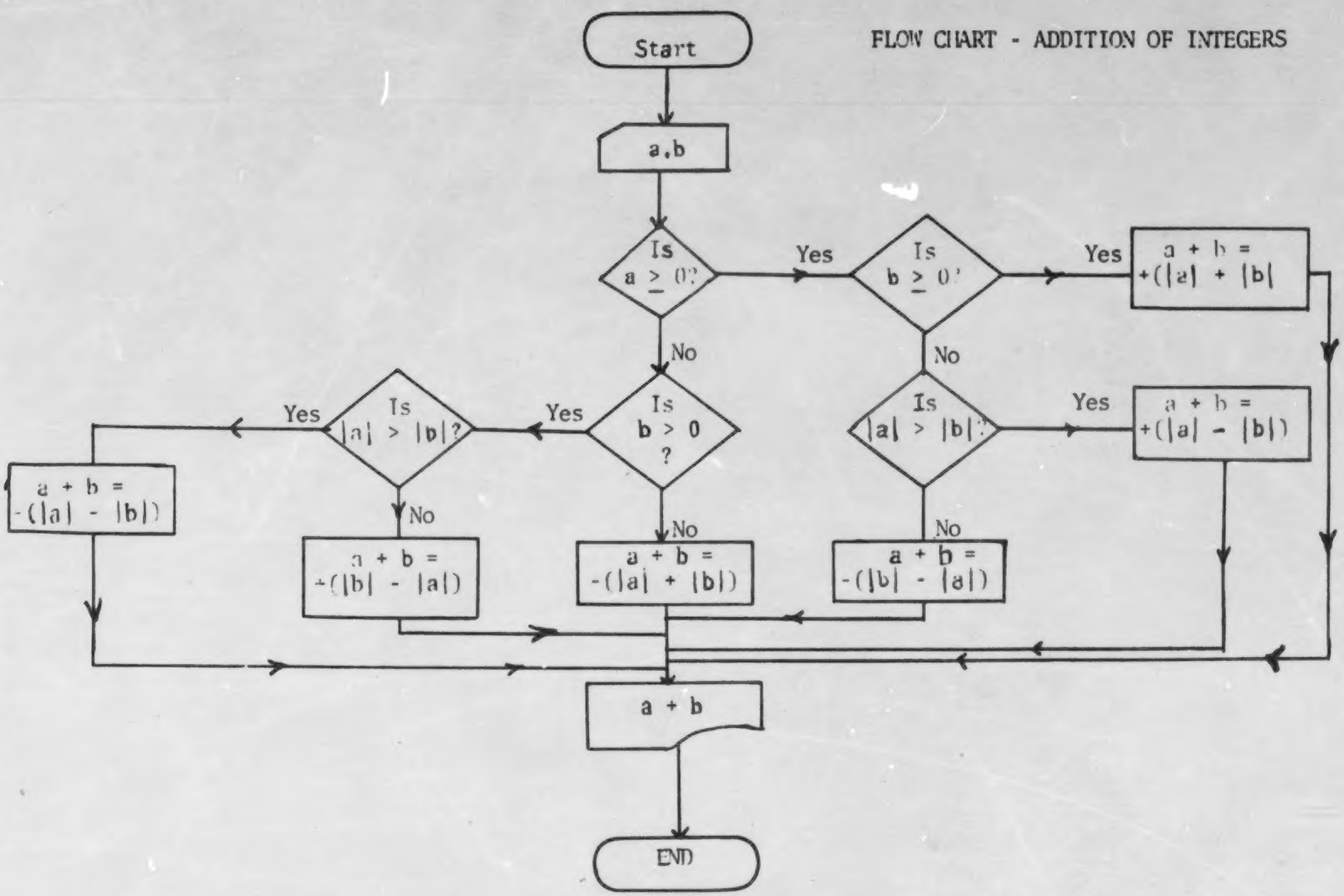
2. If $a \leq 0$ and $b \leq 0$ then $a + b = -(|a| + |b|)$

3. If $a > 0$ and $b < 0$ and $|a| \geq |b|$ then $a + b = |a| - |b|$

4. If $a \geq 0$ and $b < 0$ and $|a| < |b|$ then $a + b = -(|b| - |a|)$

5. $a - b = a + (-b)$

A flow chart of the definition of addition of integers is illustrated in Figure 1-5-15.



FLOW CHART - ADDITION OF INTEGERS

Figure 1-5-15

Problem Set 1-5-16

1. Trace through the preceding flow chart using the following pairs of elements as the numbers to be added.
    a) $+2, +3$
    b) $-5, +1$
    c) $+5, -2$
    d) $-3, -4$

2. Write a formal definition for the multiplication of two integers in terms of absolute value.

3. Prepare a flow chart for your definition of multiplication.

4. Trace through your flow chart using the following pairs of numbers as the factors to be multiplied.
    a) $+5, +2$
    b) $-3, +6$
    c) $+2, -5$
    d) $-4, -6$

Review Problem Set 1-5-17

1. Find the following sums.
    a) $5 + 2$
    b) $5 + (-2)$
    c) $(-2) + 5$
    d) $(-2) + (-5)$
    e) $(-5) + 2$
    f) $5 - (+2)$
    g) $5 - (-2)$
    h) $(-2) - (5)$
    i) $-2 - (-5)$
    j) $-7 + (-3)$
    k) $+3 + (-7)$
    l) $-7 - (+3)$
    m) $6 + 0$
    n) $8 + -10$
    o) $((-7) - 4) + 1$
    p) $(8 + (-2)) + (-3)$
    q) $((-5) + 9) + (-4)$
    r) $((-7) - (-4)) - (3)$
    s) $(+8) + (-2) - (-7)$
    t) $(-8) - (+2) - (+7)$

2. Find the following products.
    a) $(6)(-5)$
    b) $(-8)(-3)$
    c) $(2)(4)$
    d) $(5)(0)$
    e) $(-7)(1)$
    f) $(8)(-1)$
    g) $(-6)(-6)$
    h) $((-4)(5))(-3)$
    i) $(-4)((5)(-3))$
    j) $((-6)(7))(+4)$
    k) $(-6)((7)(+4))$
    l) $((-9)(-3))(-2)$

3. Find the numbers represented by the following expressions.
    a) $((3)(-4)) + (3 \cdot 4)$
    b) $(4 \cdot 5) - (4 \cdot (-2))$
    c) $4 \cdot (-3 + -7)$
    d) $(-2) \cdot (-4 - (-5))$
    e) $(+5)(-8 + (-2))$
    f) $((-12) - (-3))(-4)$
    g) $(12 + (-4))(-2)$
    h) $(-25 \cdot 4) + (-12 + (+3))$
    i) $(5 + -8) + (5 \cdot (-2))$
    j) $((-12) \cdot (-4)) + (3 \cdot (-4))$

1-6 Rational Numbers

We have already discussed the fact that numbers are concepts or ideas. The number four is a concept which we associate with a set of objects. The set may be composed of the legs on a table, the wheels on an automobile or the fingers on your left hand, not counting the thumb. In any case, the number four is an idea in our minds relative to a property which is common to all of these sets. The symbols "four", "4", and "IV" are not numbers, they are only symbols which represent a number. A number is an idea.

Somewhere in man's development, he began to feel the need to create symbols which could be used to describe the concept associated with a set of objects which had been broken or separated into parts. The set of natural numbers was perfectly suited to describing the properties of sets of "whole" objects. However, the set of natural numbers was not well suited for describing sets containing parts of objects. This idea gave rise, over a long period of time, to the set of rational numbers.

Definition 1-6-1 The Set of Rational Numbers.

The set of rational numbers Q is defined as:

$$Q = \{X \mid X = p/q,\ p \in I,\ q \in I,\ q \neq 0\}$$

This set is important for several reasons. First, it is a set on which division is defined except when the divisor is zero. Hence, (5/3) is an element of the set of rational numbers, while it is not defined if we are limited only to the set of integers. Secondly, the set of rational numbers is important because the solution set for equations of the form $ax + b = c$ may be empty if we are limited only to the integers. For example, the solution set for the equation $3x - 2 = 5$ is $\{7/3\}$. This number is an element of the set of rational numbers but is not an element of the integers. We can see that the set of rational numbers is necessary if the equation $3x - 2 = 5$ is to have a non-empty solution set.

We will now take a closer look at the set of rational numbers with the aid of the computer. Consider the following problem.

Example 1-6-2

Given the set G, a subset of the integers, $G = \{1,2,3\}$

Build the set of all possible rational number expressions which can be formed with the elements of the set G.

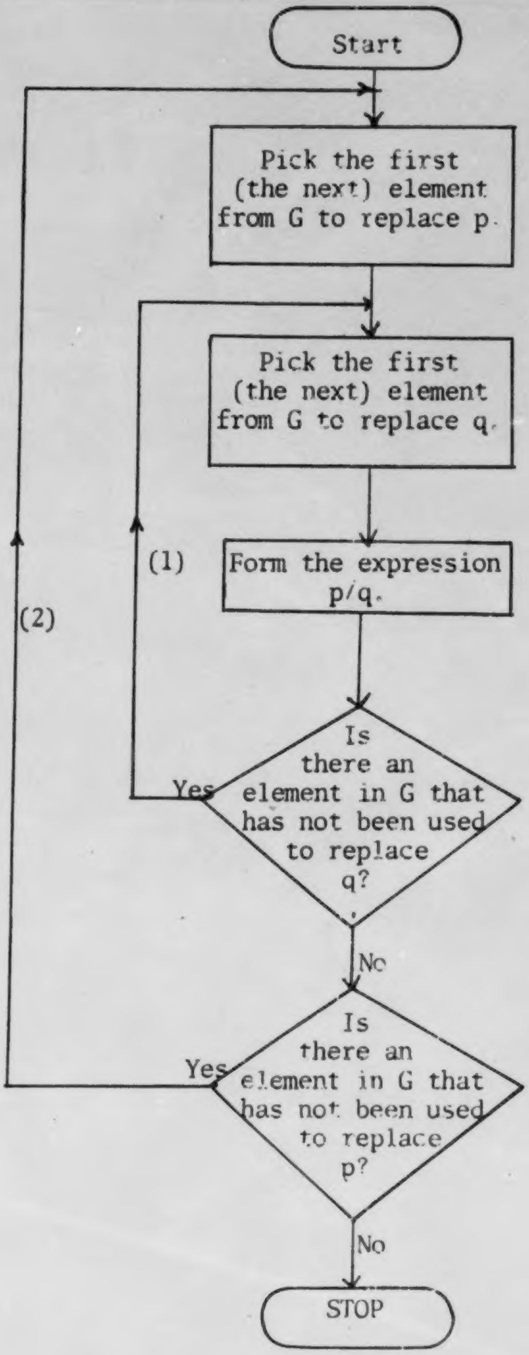

Start
Pick the first (the next) element from G to replace p.
Pick the first (the next) element from G to replace q.
Form the expression p/q.
Is there an element in G that has not been used to replace q?
Yes
(1)
No
Is there an element in G that has not been used to replace p?
Yes
(2)
No
STOP

Figure 1.6.3

As we look at the algorithm described by the flow chart in Figure 1-6-3, we see two nested loops represented by the arrows (1) and (2). Loop (1) is said to be nested inside of loop (2) because the beginning and end of loop (1) are completely enclosed within the beginning and end of loop (2). The sequence of events in the flow chart will produce the following rational expressions:

1/1,1/2,1/3,2/1,2/2,2/3,3/1,3/2,3/3.

Beginning at the (Start) and passing down the flow diagram, p = 1 and q = 1 so the first rational expression p/q is 1/1.

Since there are elements in G that have not been used to replace q, the answer to the first question is yes. Hence, we go back along loop (1) and select a second value for q. We get q = 2. Then, p = 1 and q = 2 so the second rational expression is 1/2.

Again there is an unused element in G and the answer to the first question is yes. We go back along loop (1) again and get a third replacement for q. This means that we now have p = 1 and q = 3. The third rational expression is now 1/3.

The answer to the first question is now no. We have used all elements in G as replacements for q. We go on to the second question. The answer to this question is yes. There are elements in G which have not been used to replace p. We must go back along loop (2) and get a new replacement for p. This means that p = 2. After p has been set equal to 2 we are told to pick the next value of q. Since all of the elements of G have been used as replacements for q we must start over. This means that q = 1 again. The rational expression is now 2/1.

The answer to the first question is now yes. There are elements in G which have not been used as replacements for q because we started over. Hence, we go back along loop (1) and get a new replacement for q so that p = 2 and q = 2. The rational expression is now 2/2.

The answer to the first question is still yes so we go back and get a new q. Now p = 2 and q = 3 so the rational expression becomes 2/3.

The answer to the first question is now no. We have used all of the elements in G as replacements for q since starting over. Hence, we go on to the second question. The answer to this question is yes. There is an element in G which has not been used to replace p. Hence, we go back up loop (2) for the new value of p. Now p = 3. Because all the elements of G have been used a second time as replacements for q, we must start over a third time. Hence, q = 1 and the rational expression is 3/1.

The answer to the first question is yes, so we set q = 2 and the expression is 3/2.

The answer to the first question is yes, so we set q = 3 and the expression is 3/3.

Finally, the answer to both of the questions is no. This means we have formed all possible rational expressions from the elements in the set G.

The solution to Example 1-6-2 is suggested by the following exercises.

Exercise 1-6-4

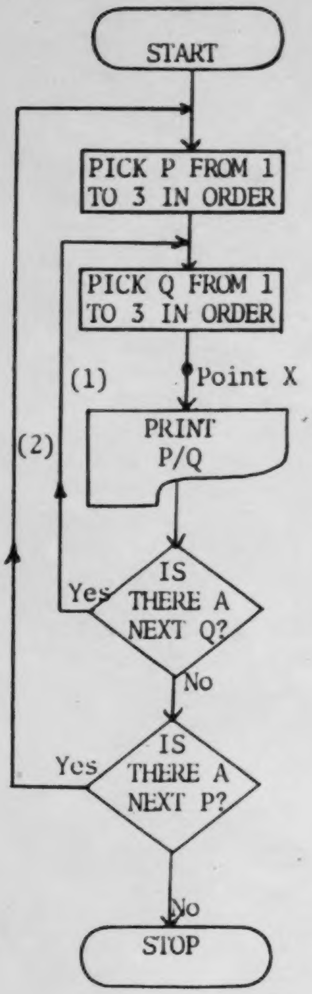


Figure 1-6-5

This flow chart shows how the diagram in Figure 1-6-3 can be made more quantitative. Use this flow chart to help you complete the table below.

Begin at the START and follow the events described by the flow chart. Each time you come to point X, STOP and fill in the chart by answering the questions at the top of each column. The first few answers have been filled in for you.

	What is the value of P?	What is the value of Q?	What loop did you just come through? (1or2)	What is the rational expression (P/Q)?	What loop will you go through next? (1 or 2)
1st time at X	1	1	none	1/1	1
2nd time at X	1	2	1	1/2	1
3rd time at X	1	3	1	1/3	2
4th time at X	2	1	2	2/1	1
5th time at X					
6th time at X					
7th time at X					
8th time at X					
9th time at X					
10th time at X					
11th time at X					
12th time at X					

Notice in Exercise 1-6-4 that values of q start over for each new value of p. Each individual expression is produced by a single pass through the inside loop. Each pass through loop (1) represents a different q. Individual expressions with equal values for p are produced by a single pass through loop (2).

We can now program the computer to produce the rational number expressions using the FOR-NEXT loops described in the previous section.

```
100 FOR P=1 TO 3

110 FOR Q=1 TO 3

120 PRINT P;"/";Q

130 NEXT Q

140 NEXT P

150 END
```

Figure 1-6-6

Notice the nested loops in this program. One pass through the P loop means three passes through the Q loop. This means that Q is changing value three times for every change in P.

Exercise 1-6-7

1. Given the set H = {1,2,3,4,5}
   What changes must be made in the program shown in Figure 1-6-6 in order to print all possible rational number expressions which can be formed with the elements of set H?
2. Write a program which will print all possible rational number expressions which can be formed with the elements in the set {-4, -3, -2, -1, 1,2,3,4}.
3. What changes in the program shown in Figure 1-6-6 must be made if the computer is to form all possible rational number expressions from the elements of the set {0,1,2,3}.
4. Write a program which will count the number of rational number expressions which can be formed with the elements of the set {1,2,3,4,5,6,7,8,9}.
5. Given a set of instructors {Jones, Smith, Henry, Adams, Parke}, a set of available rooms {Room 1, Room 13, Room 7, Room 6, Room 5}, and a set of subjects which need to be taught {Algebra, Geometry, Trigonometry, Chemistry, Physics}, draw a flow chart showing how to set up a list of all possible teacher, room, subject assignments.

In this section we reviewed the concept of a rational number as the ratio of two integers. This review served to introduce the concept of nested loops as a programming technique. Whenever two or more FOR-NEXT loops are used in programming they must either be separated completely or nested. They cannot intersect. It is a good idea for beginning programmers to practice structuring the logic of a computer program so that the loops are nested.

How does one program the computer to produce all possible rational number expressions from a set of integers which are not successive? Consider the following problem. Given the set K, a subset of the integers;

$$K = \{-1, 0, 9, -7, 5\}.$$

Build the set of all possible rational number expressions which can be formed with the elements of the set K.

Notice that the set K does not contain successive integers. In order to program the computer to solve this problem, we must introduce a new kind of machine variable, the subscripted variable.

In BASIC a single subscripted variable is written as one letter followed by a natural number in parentheses. See examples in Figure 1-6-8.

a) M(3)
b) X(24)
c) A(I)

Figure 1-6-8

The subscript is the number in the parentheses. If the parentheses contain a variable, the subscript is the number which is substituted into that variable. A subscript may be any positive integer or zero.

The program shown in Figure 1-6-9 will help you understand how subscripted variables can be used.

```
100  FOR I = 1 TO 5

110  READ A(I)

120  NEXT I

130  PRINT "WHAT IS THE SUBSCRIPT OF THE VARIABLE YOU WANT PRINTED";

140  INPUT I

150  PRINT "A(";I;") = ";A(I)
```

Figure 1-6-9 (continued on next page)

```
160  PRINT

170  GO TO 130

180  DATA -1,0,9,-7,5

190  END
```

Figure 1-6-9

The loop in lines 100-120 assigns the numbers in the DATA line to the subscripted variables A(1) through A(5). The first time through this loop the variable A(1) is set equal to the number -1. The second time through the variable A(2) is set equal to 0 and so on until all five subscripted variables have been assigned a value from the data line.

The loop in lines 130-170 is designed to let you examine the value of each subscripted variable. Notice the new BASIC instruction, INPUT I, in line 140. This instructs the computer to type a question mark and wait for the operator to enter a value for the variable I from the keyboard. The print out of this program shown in Figure 1-6-10 exhibits the relation between the subscript and the value of the variable.

```
WHAT IS THE SUBSCRIPT OF THE VARIABLE YOU WANT PRINTED?   3
A[ 3 ] = 9

WHAT IS THE SUBSCRIPT OF THE VARIABLE YOU WANT PRINTED?   1
A[ 1 ] = -1

WHAT IS THE SUBSCRIPT OF THE VARIABLE YOU WANT PRINTED?   2
A[ 2 ] = 0

WHAT IS THE SUBSCRIPT OF THE VARIABLE YOU WANT PRINTED?   5
A[ 5 ] = 5

WHAT IS THE SUBSCRIPT OF THE VARIABLE YOU WANT PRINTED?   4
A[ 4 ] = -7
```

Figure 1-6-10

The problem of programming the computer to form all possible rational number expressions from a set of integers which might not be sequential can now be solved.

Exercise 1-6-11

Given the set K, a subset of the integers,

$$K = \{-1, 0, 9, -7, 5\}$$

Use the computer to generate the set of all possible rational number expressions which can be formed with the elements from set K.

Program

```
100  FOR N = 1 TO 5
110  READ A(N)
120  NEXT N
130  DATA -1,0,9,-7,5
140  FOR I = 1 TO 5
150  FOR J = 1 TO 5
160  IF A(J) = 0 THEN 180
170  PRINT A(I); "/";A(J)
180  NEXT J
190  NEXT I
200  END
```

Output

```
-1/-1
-1/9
-1/-7
-1/5
 0/-1
 0/9
 0/-7
 0/5
 9/-1
 9/9
 9/-7
 9/5
-7/-1
-7/9
-7/-7
-7/5
 5/-1
 5/9
 5/-7
 5/5
```

Figure 1-6-12

The loop in lines 100-120 assigns the elements of set K to the subscripted variables as follows.

A(1) = -1, A(2) = 0, A(3) = 9, A(4) = -7, A(5) = 5

The nested loops in lines 140-190 generate all possible combinations of the subscripts 1 through 5 as shown below.

1 with 1
1 with 2
1 with 3
1 with 4
1 with 5
2 with 1

2 with 2
.
.
.
5 with 4
5 with 5

This allows us to form all possible combinations of the subscripted variables as follows.

A(1) with A(1)
A(1) with A(2)
A(1) with A(3)
A(1) with A(4)
A(1) with A(5)
A(2) with A(1)
A(2) with A(2)
.
.
.
A(5) with A(4)
A(5) with A(5)

Hence, we can form all possible combinations of the elements in set K into rational expressions. Line 160 is necessary because of the definition of a rational number expression. This line forces the computer to skip the PRINT instruction whenever the denominator is zero. This keeps the computer from printing expressions like (-7/0) which are not rational number expressions.

Exercise 1-6-13

1. A student is given the set {-9,4,0,13,-6} and told to program the computer to form all possible rational expressions with the integers in this set. He produces the following program.

```
100  FOR I = 5 TO 9

110  READ A(I)

120  NEXT I

130  FOR I = 5 TO 9

140  FOR J = 5 TO 9

150  IF A(J) < > 0 THEN PRINT A(I);"/";A(J)

160  NEXT J
```

Exercise 1-6-13 (continued)

```
170  NEXT I

180  END

190  DATA -9,4,0,13,-6
```

Find all of the mistakes in this program and correct the program.

2. Draw a flow chart to show the logical sequence of events necessary to find the intersection of two finite sets. Draw this flow chart with "nested" logic loops.

3. Given sets A and B.
   $A = \{1,9,-4,0,4,2,7,-9\}$
   $B = \{9,14,23,3,6,4,7\}$
   Program the computer to print out $A \cap B$.

4. Given the set $D = \{4,2,19,43,26,0,6,9,13\}$
   Find all possible rational number expressions p/q, $p \in D$, $q \in D$ such that:
   a) $0 < p/q < 1$
   b) $1 < p/q < 2$
   c) $p/q = \text{INT}(p/q)$

Fractional symbols for rational numbers are convenient for many kinds of calculations. It is common, of course, to also use decimal (base ten) numerals for the rational numbers. The numeral "1.5", for example, represents 3/2 and "$0.666\overline{6}$" represents 2/3. The bar over the 6 is to indicate that the digits are repeating endlessly in a cycle of one digit. In the numeral "$4.3232\overline{32}$," the bar indicates that the numeral repeats 32 endlessly in a cycle of two digits.

To find the decimal numeral for a rational number when a fractional symbol a/b is given, we may divide a by b. The quotient obtained will be represented by some decimal numeral which repeats. It can, in fact, be proved that all rational numbers can be named by decimal numerals that repeat endlessly; and conversely, that every decimal numeral that repeats endlessly represents a rational number. The following examples show how to find a fractional symbol for a rational number represented by a repeating decimal.

Example 1-6-14

Find a fractional symbol for $0.\overline{66}$.

Let $x = 0.\overline{66}$
Then $10x = 6.\overline{66}$
$9x = 6.\overline{00}$
$x = 6/9 = 2/3$

(These examples are good illustrations of how to find different names for a given number.)

Example 1-6-15

Find the fractional equivalent of $0.12\overline{12}$.

Let $x = 0.12\overline{12}$
Then $100x = 12.12\overline{12}$
$99x = 12.0\overline{0}$
$x = 12/99 = 4/33$

Example 1-6-16

What number is named by $0.24\overline{99}$?

Let $x = 0.249\overline{9}$
Then $100x = 24.999\overline{9}$
$99x = 24.75$
$x = 24.75/99 = 1/4$

Thus, both 0.25 and $0.249\overline{9}$ represent the same rational number, 1/4.

Example 1-6-17

Find the fractional equivalent of $2.431\overline{31}$.

Let $x = 2.431\overline{31}$
Then $100x = 243.131\overline{31}$
$99x = 240.70\overline{0}$
$x = 240.7/99 = 2407/990$

Problem Set 1-6-18

1. Do the symbols in each of the following pairs represent the same number? Explain.

   a. $\frac{3 + 4}{3}$ and $1 + 4$ b. $\frac{6 + 12}{6}$ and $1 + 2$

   c. $\frac{4 + 8}{12}$ and $\frac{1 + 2}{3}$ d. $\frac{4 - 6}{2}$ and $\frac{2(2 - 3)}{2}$

2. Find decimal numerals for these rational numbers:

   a. $\frac{1}{7}$ b. $\frac{3}{75}$

   c. $\frac{4}{11}$ d. $\frac{4}{15}$

Problem Set 1-6-18 (continued)

3. Write a fractional numeral for each of the following numbers:

a. $0.\overline{22}$ c. $0.24\overline{24}$ e. $1.42\overline{42}$

b. $0.13\overline{13}$ d. $0.0\overline{55}$ f. $3.142\overline{142}$

4. Show that $\frac{10}{11} = 0.90\overline{90}$.

5. Show that $0.23678\overline{78}$ is a rational number.

*6. Given the set of digits {0,1,2,3,4,5,6,7,8,9}
Program the computer to print out the decimal equivalent for each different rational number which can be formed with these digits. Print out the rational expression (in lowest terms) for this number and determine how many different numbers were printed.

7. Determine the out put from the following program without using the computer.

```
100  FOR I = 1 TO 2 STEP 0.5

110  FOR J = 1 TO 3

120  FOR K = 1 TO 2

130  PRINT I;J;K;

140  NEXT K

150  PRINT

160  NEXT J

170  NEXT I

180  END
```

8. Write a program to produce the following output.

```
1  1  1  1  1  2

1  2  1  1  2  2

1  3  1  1  3  2

2  1  1  2  1  2

2  2  1  2  2  2

2  3  1  2  3  2
```

Problem Set 1-6-18 (Continued)

*9. What is the maximum number of digits in the repeating cycle of the decimal representation for a rational number $p/q, q \neq 0$.

Review Problem Set 1-6-19

Evaluate the following expressions. Write your answers in rational form. Reduce all fractions to lowest terms.

1. $\frac{1}{2} + \frac{1}{3}$
2. $\frac{2}{3} + (-\frac{1}{6})$
3. $\frac{1}{8} - \frac{1}{4}$
4. $\frac{1}{3} - (-\frac{3}{10})$
5. $-\frac{1}{4} - \frac{1}{6}$
6. $-\frac{3}{6} - \frac{-1}{2}$
7. $\frac{3}{6} \cdot \frac{4}{9}$
8. $\frac{3}{8} \cdot \frac{-1}{6}$
9. $\frac{-3}{1} \cdot \frac{2}{-3}$
10. $\frac{7}{6} \cdot (-(\frac{-1}{9}))$
11. $(\frac{1}{2} \cdot \frac{1}{4}) \cdot \frac{3}{8}$
12. $(\frac{1}{2})(\frac{3}{4})(-\frac{1}{12})$
13. $\frac{9 - 6}{10 - 5}$
14. $\frac{9}{10} - \frac{6}{5}$
15. $\frac{7 + 3}{15 - 5} - \frac{52}{26 \cdot 2}$
16. $(\frac{1}{50} + \frac{3}{25}) + (\frac{2}{5} + \frac{3}{50})$
17. $(\frac{60}{5} + \frac{30}{5})\frac{0}{18}$
18. $\frac{2}{3} - \frac{58}{87}$
19. $\frac{3 \times 978}{5 \times 978}$
20. $(90 \times \frac{4}{9})$
21. $\frac{\frac{2}{3} + \frac{3}{4}}{9} - \frac{1}{5}$
22. $\frac{\frac{1}{4}}{\frac{-2}{3} + \frac{4}{6}} - 1$
23. $\frac{(\frac{2}{3} + \frac{3}{4})}{6} \times \frac{5}{9}$
24. $\frac{\frac{2}{3}}{\frac{3}{2}} + (-1)$

1-7 Irrational Numbers

You learned in geometry that a number may be used as the measure of a line segment. Consider the right triangle in Figure 1-7-1. The legs of this triangle have the measure one.

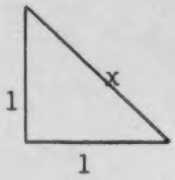


Figure 1-7-1

Obviously there will be a numerical representation for the measure of the hypotenuse of this triangle. Since it is unknown, we will call it x for the time being. Can we find this number?

Recalling the Pythagorean Theorem from geometry we know that:

$$x^2 = 1^2 + 1^2$$

or

$$x^2 = 2$$

This means that the number representing the length of the hypotenuse must be a solution for this equation. You can see by inspection that there are no integral solutions for this equation. Hence, we will look for our solution in the set of rational numbers.

If the solution to the equation $x^2 = 2$ is a rational number, then $x = (p/q)$, $q \neq 0$, $p \varepsilon I$ with p and q relatively prime. That is p and q contain no common factors.

Then $(p/q)^2 = 2$ $\qquad q \neq 0, q \varepsilon I, p \varepsilon I$

or $p^2 = 2q^2$.

Since $q \varepsilon I$, $2q^2$ is an even integer. Hence, p^2 must also be an even integer. If p^2 is an even integer, it can be proved that p is also an even integer. Therefore, p can be expressed as 2r where $r \varepsilon I$.

Then $(2r)^2 = 2q^2$

and $2r^2 = q^2$.

By the same type of argument we can see that q must also be an even integer. We have now shown that p and q are both even. Hence, they contain the common factor 2. This contradicts our initial assumption that they were relatively prime.

Hence, we conclude that $x \notin Q$.

This is an example of indirect proof, a very valuable tool in mathematics. You will not be required to develop proofs of this nature at this time. However, you should recognize that we have established a need for an additional set of numbers, since the solution to the equation $x^2 = 2$ is not in the set of rational numbers. The solution to this equation will be named by the symbol "$\sqrt{2}$". This symbol "$\sqrt{2}$" represents that positive number which when multiplied by itself yields 2. It is called an irrational number.

All decimal representations of rational numbers are repeating decimals such as $3/8 = 0.375\overline{0}$ and $5/11 = 0.4545\overline{45}$. The irrational numbers cannot be expressed in repeating decimal form. If they could be expressed as repeating decimals they could then be written as rational numbers using the algorithm shown in Section 1-6. This set of all non-repeating decimals is called the set of irrational numbers Ƶ.

An interesting problem is to find rational number expressions which are good approximations to irrational numbers.

Example 1-7-2

Find a rational number p/q, $q \neq 0$, such that

$$|(p/q)^2 - 2| < 0.001.$$

This problem is directing us to find a rational number p/q, $q \neq 0$, which when squared will be within 0.001 of the number two. The number p/q, $q \neq 0$, will then be a rational approximation for the irrational number $\sqrt{2}$. In this example the number 0.001 is referred to as the tolerance. An algorithm for finding this rational approximation is illustrated in the following flow chart.

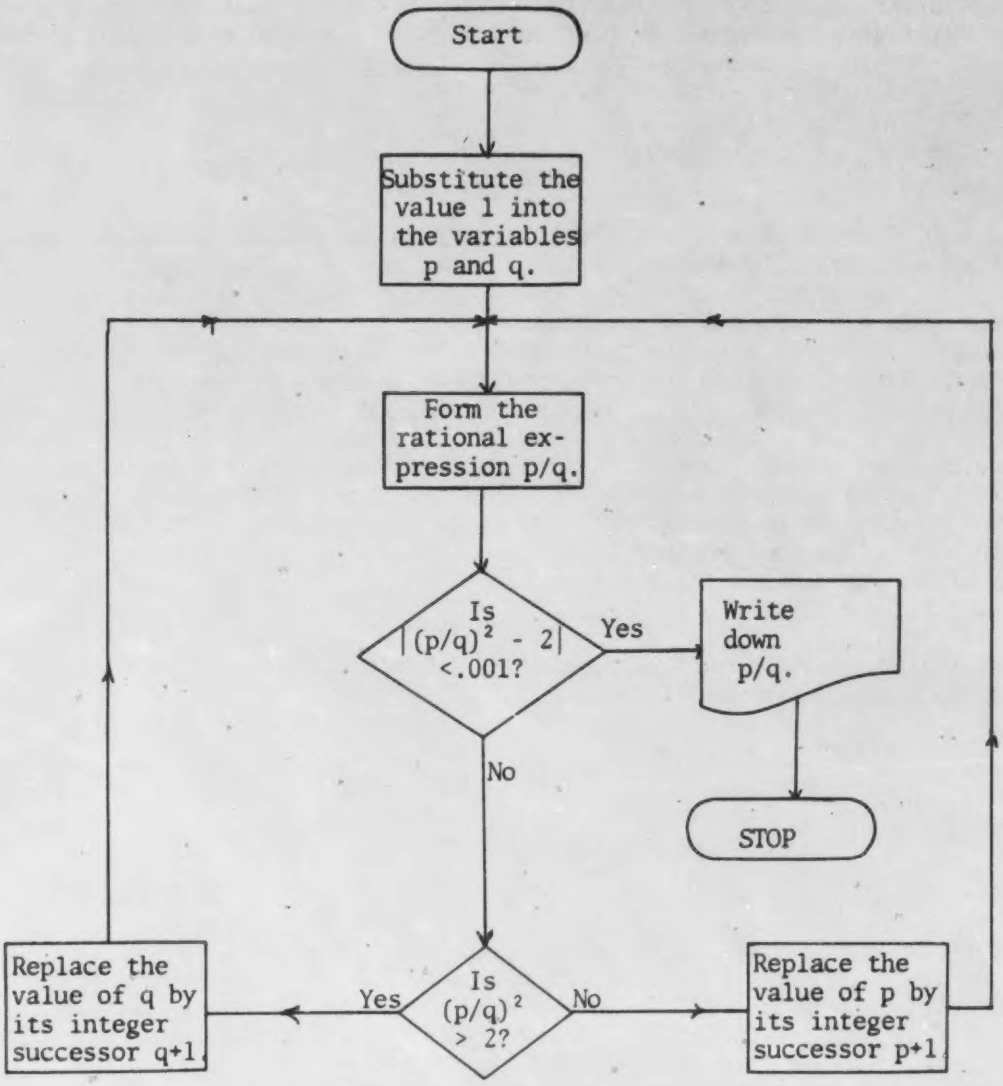

Start
Substitute the value 1 into the variables p and q.
Form the rational expression p/q.
Is |(p/q)² - 2| <.001?
Yes
Write down p/q.
STOP
No
Is (p/q)² > 2?
Yes
No
Replace the value of q by its integer successor q+1.
Replace the value of p by its integer successor p+1.

Figure 1-7-3

The algorithm shown in Figure 1-7-3 will produce the following rational approximations to $\sqrt{2}$.

p/q Rational Form	p/q Decimal Form	$(p/q)^2$ Square of the fraction
1/1	1	1
2/1	2	4
2/2	1	1
3/2	1.5	2.25
3/3	1	1
4/3	1.333	1.55
5/3	1.666	2.77
5/4	...	...
178/126	1.4127	1.99572
179/126	1.42063	2.01820
179/127	1.40945	1.98655
180/127	1.41732	2.00880
180/128	1.40625	1.97754
181/128	1.41406	1.99957

From the last entry in this table you can see that;

$$|(p/q)^2 - 2| = |1.99957 - 2| = 0.00043 < 0.001.$$

So a rational approximation to $\sqrt{2}$ which is within the specified tolerance, 0.001, is 181/128.

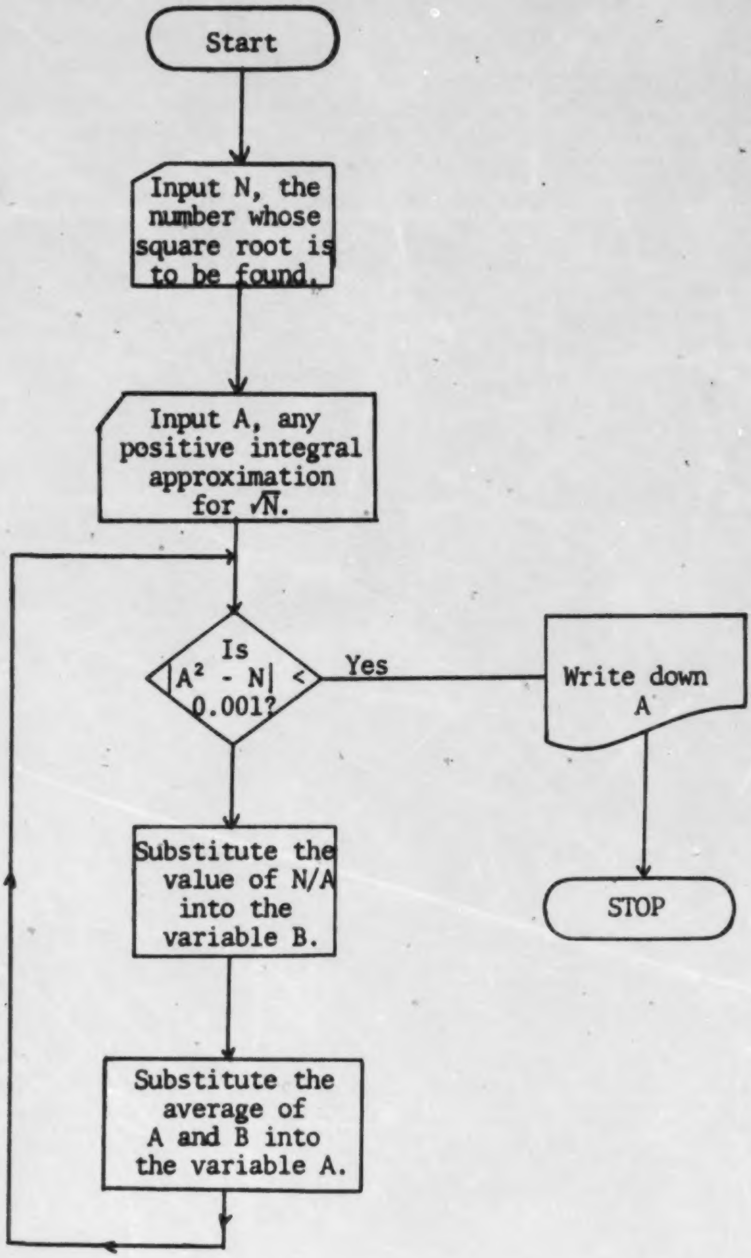
Exercise 1-7-4

Write and execute a BASIC program to find a rational expression approximating $\sqrt{a}$, $a > 0$ with a tolerance of 0.0001. Base your program on the algorithm described in Example 1-7-2.

It may be helpful in writing this program to know that any algebraic expression of the form $|x|$ may be written in BASIC as ABS(X).

A faster method of approximating the square root of a number is one known as Newton's Method. The algorithm for this method is shown in the following example.

Example 1-7-5


Start
Input N, the number whose square root is to be found.
Input A, any positive integral approximation for $\sqrt{N}$.
Is $|A^2 - N| <$ 0.001?
Yes
Write down A
STOP
Substitute the value of N/A into the variable B.
Substitute the average of A and B into the variable A.

Exercise 1-7-6

1. Follow the algorithm in Example 1-7-5 without the use of the computer, and find $\sqrt{6.76}$ with a zero tolerance. Carry 3 decimal places in your calculations.

2. Follow the algorithm in Example 1-7-5 and, without the use of the computer, find an approximation for $\sqrt{14}$ with a tolerance of 0.01. Carry no more than 3 decimal places in your calculation.

3. Program the computer to find an approximation to the square root of any positive number by Newton's Method with a tolerance of 0.0001. Use the program to print out the square root of the first ten natural numbers.

4. Any expression of the form $\sqrt{x}$ can be represented in BASIC by the expression SQR(X). Whenever this expression is used, the computer will substitute a rational approximation for the square root of the number X. Use this expression SQR(X) and the computer to produce the square roots of the first 15 natural numbers. Compare the results with those from problems 2 and 3.

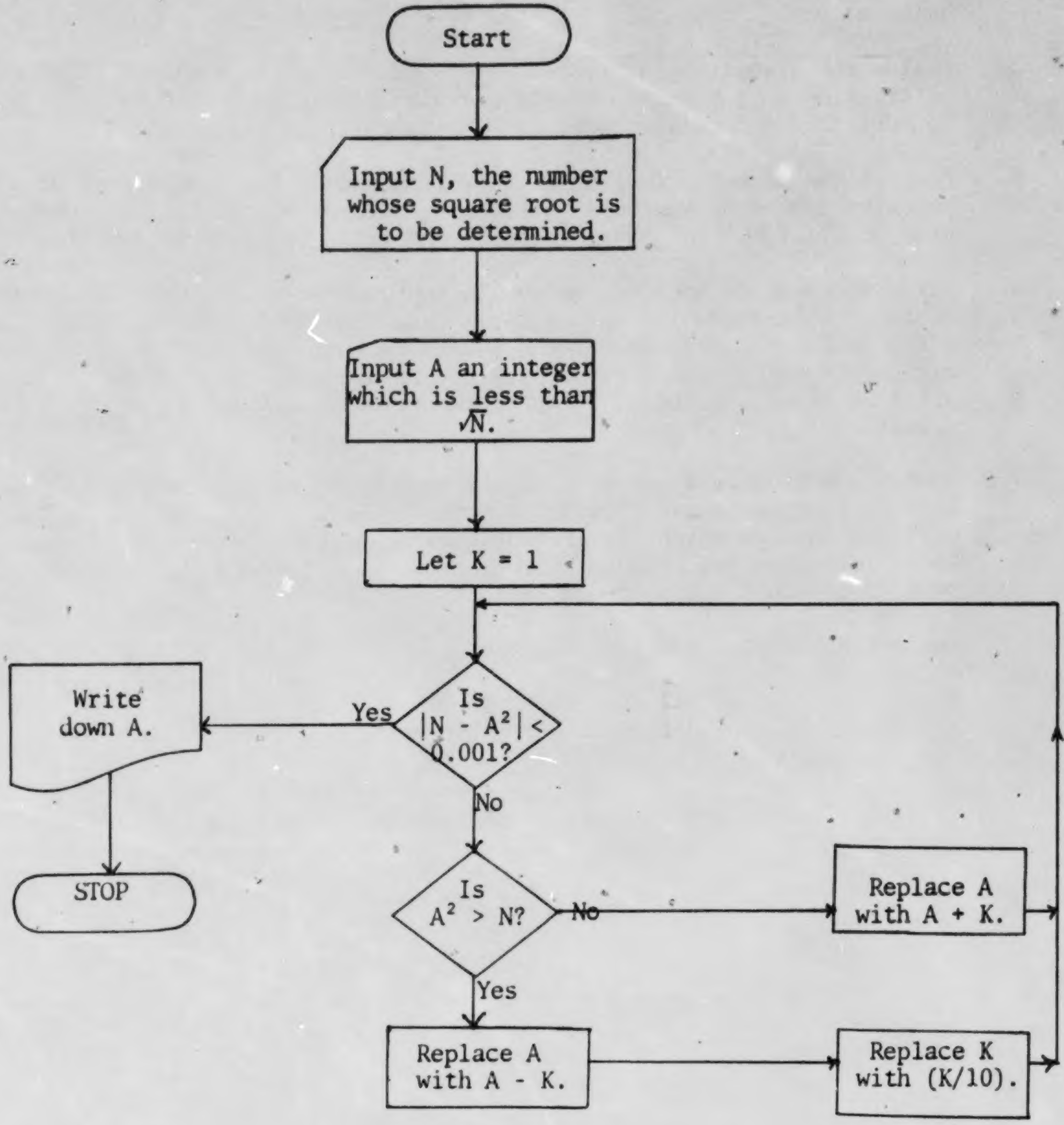
5. Newton's Method is a powerful algorithm and can be used effectively by hand to find the square root of a number. This method depends on a property of numbers which states that given any positive number, which can be expressed as the product of two unequal positive factors, the square root of the number is between these two factors.

   Explain why Newton's Method works.

A third method for determining the square root of any positive number is shown in the following Problem set 1-7-7.

Problem Set 1-7-7

1. Follow the algorithm shown below and determine the square root of 10.4976.



2. Program this algorithm into the computer and test it by printing out the square roots of the first 10 natural numbers.

In this section we have reviewed the concept of an irrational number. In addition, we used the computer to find rational approximations for irrational numbers. In first year algebra you learned that many irrational numbers can be named precisely by use of the radical sign, thus, avoiding the problem of approximation. Thus, 1.414 may be a good approximation for the number which when squared yields 2. However, this number is described exactly only be a name involving the radical sign such as $\sqrt{2}$. In general, the symbol $\sqrt{a}$, $a \geq 0$, will represent that **positive number whose square the number a.**

There are also two properties for the simplification of radicals which you should remember from first year algebra. They are used to simplify the names of irrational numbers involving the radical sign. They are: $\sqrt{a}$ $\sqrt{b}$ $a \geq 0$, $b \geq 0$, $a, b, \in Q$ or Z

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

and

$$\sqrt{a/b} = \sqrt{a}/\sqrt{b}$$

Review Problem Set 1-7-8

1. Evaluate each of the following.

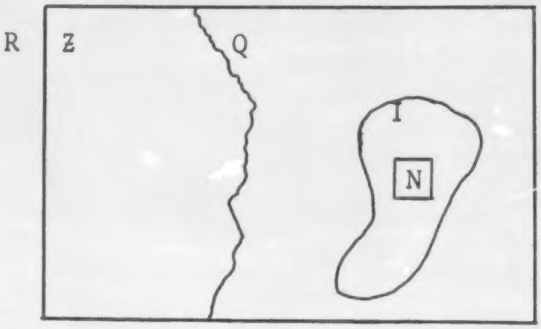
   a. $\sqrt{16}$
   b. $\sqrt{81}$
   c. $\sqrt{169/625}$
   d. $\sqrt{75}$
   e. $-\sqrt{64}$
   f. $\sqrt{160}$
   g. $\sqrt{52}$
   h. $\sqrt{9 + 16}$
   i. $\sqrt{9} + \sqrt{16}$

2. Simplify each of the following expressions.

   a. $\sqrt{2xy} \cdot \sqrt{8x}$
   b. $\sqrt{300}$
   c. $\sqrt{16x^2} \cdot \sqrt{y^3}$
   d. $-\sqrt{8x^6}$
   e. $-\sqrt{160}$
   f. $-\sqrt{x^3}$
   g. $\sqrt{x^2} + \sqrt{4x^2}$
   h. $\sqrt{3} \cdot \sqrt{27}$
   i. $\sqrt{-x^3}$

1-8 The Real Numbers

So far, the natural numbers, integers, rational numbers and irrational numbers have been discussed. The natural numbers (N) form a proper subset of the integers (I). The integers form a proper subset of the rational numbers (Q). The rational numbers (Q) and the irrational numbers (Z) are non-intersecting sets each of which is a proper subset of a set of numbers which are called the real numbers. The set of real numbers (R) is the union of the set of rational numbers (Q) and the set of irrational numbers (Z). Unless otherwise stated the replacement set for any variable will be assumed to be from the set of real numbers.



$N \subseteq I$
$I \subseteq Q$
$Q \subseteq R$
$Z \subseteq R$
$Q \cup Z = R$
$Q \cap Z = \emptyset$
$N \subseteq Q$
$N \subseteq R$
$N \cap Z = \emptyset$
$I \cap N = N$
$I \cup N = I$
$Q \cap I = I$

Figure 1-8-1

Exercise 1-8-2

1. Given the sets of numbers N,I,Q,Z,R and the set A

$$A = \{-\sqrt{2}, \sqrt{9}, \frac{\sqrt{2}}{3-3}, \frac{0}{8}, \frac{0}{0}, 4, \sqrt{3.14}, \frac{\sqrt{5}}{\sqrt{5}}, \frac{63}{7}, \frac{5}{4}, -\sqrt{36}, 5+\frac{1}{8}, \frac{\sqrt{9}}{\sqrt{64}}, \frac{-6}{0}, \sqrt{-8}\}$$

List the elements in each of the following sets.

a. $N \cap A$
b. $I \cap A$
c. $Q \cap A$
d. $Z \cap A$
e. $R \cap A$
f. $(A \cap N) \cup (A \cap I)$
g. $(A \cap Z) \cup (A \cap Q)$
h. $A \cap \{0\}$

We have reviewed the set of real numbers and described in detail other sets of numbers which are proper subsets of the real numbers. We have also reviewed the operations of addition and multiplication on the set of integers. As you know, these definitions are applicable to the set of real numbers as well.

The set of real numbers, along with the definitions of addition and multiplication, form a mathematical system called the real number system. The properties of this system will be developed formally in Chapter 2. In the meantime there is one problem remaining to be discussed. That is the problem of the order of mathematical operations within an expression.

Given the expression $3 + 4 \cdot 5$, there are two possible values for this expression depending on which operation is carried out first. The order of operations can be indicated by parentheses. That is $(3 + 4) \cdot 5 = 35$ while $3 + (4 \cdot 5) = 23$. Whenever more than one set of grouping symbols are used, the operation in the innermost set of grouping symbols is performed first. Hence,

$$\begin{aligned} 3 \cdot (7 - (6 + 2)) &= 3 \cdot (7 - 8) \\ &= 3 \cdot (-1) \\ &= -3 \end{aligned}$$

This same procedure is also followed by the computer.

Any desired order of operations may be indicated with the aid of parentheses.

Mathematical expressions can become very complex and cumbersome if all symbols of grouping are included. For this reason agreements have been made for omiting grouping symbols. These agreements are stated below and are followed precisely by the computer.

First:	Perform exponentiations.
Second:	Perform oppositing.
Third:	Perform multiplications and divisions in left to right order.
Fourth:	Perform additions and subtractions in left to right order.

For example:	$-2 \cdot 3^2 + 4 \cdot -5 + -(-72) \div (6 \text{ x } 6)$
First:	$-2 \cdot (3^2) + 4 \cdot -5 + -(-72) \div (6 \text{ x } 6)$
Second:	$-2 \cdot (3^2) + 4 \cdot -5 + [-(-72)] \div (6 \text{ x } 6)$
Third:	$[-2 \cdot (3^2)] + (4 \cdot -5) + \{[-(-72)] \div (6 \text{ x } 6)\}$
Fourth:	$\{[-2 \cdot (3^2)] + (4 \cdot -5)\} + \{[-(-72)] \div (6 \text{ x } 6)\}$

As you can see, these agreements for omiting grouping symbols greatly simplify the writing of mathematical expressions. They also make it easier to write expressions in BASIC.

Exercise 1-8-3

1. Insert grouping symbols in the following expressions which indicate the order of operations as described in the agreements above and then simplify each expression by performing the indicated operations.

a. $3 \times 2 - 4 \div 2$ b. $6 - 3 + 2 - 4 \times 2 \div 2$

c. $5^2 + -3^2 - 2 \times 4$ d. $3 \times 4^2 + 2 - 6 \times 5$

e. $1 + 2 \times 3 - 4 \div 5$ f. $29 \times \frac{1}{2} + 3 \div 9$

2. Write the following expressions in an equivalent form using the least possible number of grouping symbols.

a. $\{((2 + 3) \times (4 - 9)) \div (17 + 1)^2 \times 2\}$

b. $(2)^4 + (7 \times 2) - 16 + (1 \div 4)$

c. $\{[(2 \cdot 3)^2 + (4 \div 2)] - 4\} + [(8 - 7) \cdot 13]$

Chapter 2

The Real Number System

2-1 The Field of Real Numbers.

In this chapter we will study the real numbers as a mathematical system. The basic requirements for a mathematical system are a set of elements, one or more operations defined on these elements, and a set of properties that are generalizations about these elements and operations. In Chapter 1 we described the set of real numbers R. This will be our set of elements. The operations on this set will be the binary operations of addition, +, and multiplication, ·, which were also reviewed in the first chapter. We will now turn our attention to the properties of this mathematical system, denoted $(R,+,\cdot)$.

Property 2-1-1 Closure for Addition (ClA)

$\forall a \in R, \forall b \in R$, there exists a unique sum $(a + b) \in R$.

This closure property for addition states that the sum of two real numbers is always a unique real number.

Property 2-1-2 Associative Property for Addition (APA)

$\forall a \in R, \forall b \in R, \forall c \in R, (a + b) + c = a + (b + c)$

As an example, $(2 + 3) + (-7) = 2 + (3 + (-7))$

Property 2-1-3 Additive Identity (AId)

$\forall a \in R$, there exists a real number, 0, such that $0 + a = a + 0 = a$.

For example, $0 + 3 = 3 + 0 = 3$.

Property 2-1-4 Additive Inverse (AIn)

$\forall a \in R$, there exists $-a \in R$ such that $a + -a = -a + a = 0$

For example, $\frac{13}{17} + -\frac{13}{17} = -\frac{13}{17} + \frac{13}{17} = 0$

Property 2-1-5 Commutative Property for Addition (CPA)

$\forall a \in R, \forall b \in R, a + b = b + a$

For instance, $113 + -17 = -17 + 113$.

Property 2-1-6 Closure for Multiplication (ClM)

$\forall a \in R, \forall b \in R$, there exists a unique product $(a \cdot b) \in R$.

This closure property for multiplication states that the product of two real numbers is always a unique real number.

Property 2-1-7 Associative Property for Multiplication (APM)

$\forall a \in R, \forall b \in R, \forall c \in R, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

As an example, $((1/3) \cdot 6) \cdot 7 = (1/3) \cdot (6 \cdot 7)$.

Property 2-1-8 Multiplicative Identity (MId)

$\forall a \in R$, there exists a real number 1, $1 \neq 0$, such that $1 \cdot a = a \cdot 1 = a$.

For instance $1 \cdot 23 = 23 \cdot 1 = 23$.

Property 2-1-9 Multiplicative Inverse (MIn)

$\forall a \in R, a \neq 0$, there exists $\frac{1}{a} \in R$ such that $\frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1$.

For example, $\frac{1}{\frac{2}{3}} \cdot \frac{2}{3} = \frac{2}{3} \cdot \frac{1}{\frac{2}{3}} = 1$

Property 2-1-10 Commutative Property for Multiplication (CPM)

$\forall a \in R, \forall b \in R, a \cdot b = b \cdot a$

As an example, $\frac{1}{3} \cdot \frac{6}{7} = \frac{6}{7} \cdot \frac{1}{3}$.

Property 2-1-11 Distributive Property of Multiplication over Addition (DMA)

$\forall a \in R, \forall b \in R, \forall c \in R, a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

As an instance, $\frac{2}{7}(-2 + 16) = \frac{2}{7} \cdot (-2) + \frac{2}{7} \cdot (16)$.

Problem Set 2-1-12

1. For each of the following, state whether the two given numbers are the additive inverses for each other. Explain why or why not.

   a. -0.001, 1.001

   b. 0, 0

   c. $\frac{1}{500}$, -0.002

   d. $(-1/2)^3$, $-(1/2)^3$

   e. $3 + \sqrt{2}$, $3 - \sqrt{2}$

   f. 2, $-1/2$

   g. $\frac{1}{3}$, $\frac{-1}{-3}$

2. For each of the following, state whether the two given numbers are multiplicative inverses of each other. Explain why or why not.

   a. -5, -0.02

   b. $\frac{2}{3}$, $-\frac{3}{2}$

   c. -3, $-\frac{3}{3}$

   d. $3 + 2\sqrt{2}$, $3 - 2\sqrt{2}$

   e. -1, -1

   f. $(-2)^3$, $-(\frac{1}{2})^3$

   g. $\frac{1}{1 - \frac{1}{2}}$, $\frac{1}{2}$

The set A where $A = \{a + b\sqrt{2} \mid a, b \text{ are rational}\}$ together with the operations of addition and multiplication forms a mathematical system $(A, +, \cdot)$ within the system of real numbers. Problem 3 through 9 are concerned with this subsystem $(A, +, \cdot)$.

Problem Set 2-1-12 (continued)

3. Perform the indicated operations and express the results in the form $a + b\sqrt{2}$.

    a. $(5\sqrt{2})(3 + \sqrt{2})$

    b. $(4 + 2\sqrt{2})(2 + 3\sqrt{2})$

    c. $(-1 + 6\sqrt{2})^2$

    d. $(1 + \sqrt{2})^3$

    e. $(10 + 6\sqrt{2}) + (2 + \sqrt{2})^2$

    f. $(-3 + 2\sqrt{2})^2 + (15 - 11\sqrt{2})$

    g. $(a + b\sqrt{2}) + (c + d\sqrt{2})$

4. a. Is there an identity element for addition in $(A, +, \cdot)$? If so, what is it?

    b. Is there an identity element for multiplication in this system? If so, what is it?

    c. What is the additive inverse of $a + b\sqrt{2}$?

5. Given $(a + b\sqrt{2}) \in A$, form the multiplicative inverse of $(a + b\sqrt{2})$ and show that this multiplicative inverse is an element of A.

6. Each of the following statements is an instance of one of the properties 2-1-1 through 2-1-11. State the abreviation for the property of which each of the following is an instance,

$$\forall x \in R, \forall y \in R, \forall z \in R$$

    a. $(3x + 2)(2x + 4) = (3x + 2) \cdot 2x + (3x + 2) \cdot 4$

    b. $(x + y) \cdot \frac{1}{x + y} = 1, x + y \neq 0$

    c. $3x + 2x = (3 + 2)x$

    d. $\pi + -\pi = 0$

    e. $\frac{x}{3} \cdot \frac{1}{\frac{x}{3}} = 1, x \neq 0$

    f. $x \cdot y \in R$

    g. $(a + -a) + 0 = 0$

    h. $[(x + -2) + (y + 4)] + (z + 2) = (x + -2) + [(y + 4) + (z + 2)]$

i) $[(-8 + x) + (y + 6)] \cdot (8 + z) = (8 + x) \cdot (8 + z) + (y + 6) \cdot (8 + z)$

j) $(x + y) \cdot (z + 4) = (z + 4)\ (x + y)$

k) $(z + 0) \cdot x = x \cdot (z + 0)$

2-2 Field Properties.

We have discussed the set of real numbers, and the properties associated with the operations of addition and multiplication. In doing so we have acquainted you with a mathematical system which is called a field. We will now formally define a field.

Definition 2-2-1 Definition of a Field.

A system denoted by (F, #, $) consisting of a set of elements F together with two binary operations # and $ is a field, if and only if

1. Closure property for # - VaεF, VbεF, (a # b)εF.
2. Associative property for # - VaεF, VbεF, VcεF, (a # b) # c = a # (b # c).
3. Identity property for # - There exists an element i ε F such that, VaεF, i # a = a # i = a.
4. Inverse property for # - VaεF, there exists an element $a^{\square}$ ε F such that $a^{\square}$ # a = a # $a^{\square}$ = i.
5. Commutative property for # - For VaεF, VbεF, a # b = b # a.
6. Closure property for $ - VaεF, VbεF, VcεF, (a $ b) ε F.
7. Associative property for $ - VaεF, VbεF, VcεF, (a $ b) $ c = a $ (b $ c).
8. Identity property for $ - There exists an element i' ε F, with i' ≠ i such that, VaεF, i' $ a = a $ i' = a.
9. Inverse property for $ - VaεF, where a ≠ i, there exists an element a^{Δ} ε F such that a^{Δ} $ a = a $ a^{Δ} = i'.
10. Commutative property for $ - VaεF, VbεF, a $ b = b $ a.
11. Distributive property - VaεF, VbεF, VcεF, a $ (b # c) = (a $ b) # (a $ c) and (a # b) $ c = (a $ c) # (b $ c).

If a system, consisting of a set of elements F together with two binary operations # and $ on F, satisfies this definition, we say the system, (F, #, $), is a field. Careful examination of definition 2-2-1 reveals that the real number system $(R, +, \cdot)$ is a field,

if F represents the set of real numbers, R,

# represents the operation of addition, +,

$ represents the operation of multiplication, ·,

i represents the additive identity, 0,

$a^{\square}$ represents the additive inverse of a, -a,

i' represents the multiplicative identity, 1, and

a^{Δ} represents the multiplicative inverse of a, $\frac{1}{a}$.

You should carefully note that F and (F, #, $) are different things. F is just the set of elements under discussion, while (F, #, $) is a mathematical system. Once the distinction is clear to you, we will refer to (F, #, $) as "the system F". For example, we commonly refer to the system of real numbers R rather than use the notation $(R, +, \cdot)$.

Exercise 2-2-2

Give a counter example which demonstrates why each of the following systems is not a field.

1. (A, #, $) where $A = \{x | x = 2n - 1, n \varepsilon I, x \varepsilon R\}$
   # is real number addition
   $ is real number multiplication

2. (B, #, $) where $B = \{x | x = 2n, n \varepsilon I, x \varepsilon R\}$
   # represents real number addition
   $ represents real number multiplication

3. (C, #, $) where C = R
   and $\forall a \varepsilon C, \forall b \varepsilon C$
   $a \# b = a + 2 \cdot b$
   $a \$ b = a - 2 \cdot b$

4. (D, #, $) where $D = \{0, 1\}$
   and $\forall a \varepsilon D, \forall b \varepsilon D$
   $a \# b = a + b$
   $a \$ b = a \cdot b$

Exercise 2-2-2 (continued)

5. $(E, \#, \$)$ where $E = \{0, 1, -1\}$
and $\forall a \in E, \forall b \in E$
$a \# b = a + (-a)$
$a \$ b = a \cdot b$

In Exercise 2-2-2 you demonstrated that certain mathematical systems, containing infinite sets of elements, are not fields. This was done by finding counter examples that were specific instances in which one or more of the field properties were not true. If we try the same technique on the systems $(R, +, \cdot)$ and $(Q, +, \cdot)$, we can find no such counter examples. Does this prove that these systems are fields? Certainly not. One method to establish that a system is a field is to show that all eleven of the properties hold for every possible substitution of elements from the set. Since the systems $(R, +, \cdot)$ and $(Q, +, \cdot)$ contain infinite sets, this substitution is an impossible task. Therefore, the eleven field properties are assumed to be valid in the system $(R, +, \cdot)$. All additional properties of real numbers, such as the addition of fractions, can be proved on the basis of this assumption. This procedure will be demonstrated in Section 2-3.

In order to extend your understanding of the field properties we will concentrate on systems involving finite sets and arbitrary binary operations. There are two reasons for studying finite systems. First, these systems involving finite sets may play a large part in your future work in mathematics. Secondly, you can use the computer to determine whether or not these finite systems are fields.

Example 2-2-3

Given the set $A = \{0, 1\}$
and the binary operations # and $ defined by the tables below.

#	0	1
0	0	1
1	1	0

$	0	1
0	0	0
1	0	1

Is the system $(A, \#, \$)$ a field?

a) We can see at a glance that the set A is closed with respect to both operations # and $ since no elements outside of set A appear in the operation tables. Hence, both closure properties are satisfied.

b) In order to check the associative property for both operations, all possible substitutions of the elements of A must be made in the expression $(a \# b) \# c = a \# (b \# c)$ and $(a \$ b) \$ c = a \$ (b \$ c)$.

A list of all possible substitutions of 0 and 1 into a, b, and c is shown in the table below.

a	b	c
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Table 2-2-4

The truth value for each of these substitutions must be checked. If each substitution yields a true statement, the operations are associative. This procedure is demonstrated below.

Check of

$\underline{(a \# b) \# c = a \# (b \# c)}$

$(0 \# 0) \# 0 \overset{?}{=} 0 \# (0 \# 0)$

$0 \# 0 \overset{?}{=} 0 \# 0$

$0 = 0$ True

$(0 \# 0) \# 1 \overset{?}{=} 0 \# (0 \# 1)$

$0 \# 1 \overset{?}{=} 0 \# 1$

$1 = 1$ True

$(0 \# 1) \# 0 \overset{?}{=} 0 \# (1 \# 0)$

$1 \# 0 \overset{?}{=} 0 \# 1$

$1 = 1$ True

. . .

Check of

$\underline{(a \$ b) \$ c = a \$ (b \$ c)}$

$(0 \$ 0) \$ 0 \overset{?}{=} 0 \$ (0 \$ 0)$

$0 \$ 0 = 0 \$ 0$

$0 = 0$ True

$(0 \$ 0) \$ 1 \overset{?}{=} 0 \$ (0 \$ 1)$

$0 \$ 1 \overset{?}{=} 0 \$ 0$

$0 = 0$ True

. . .

. . . . . .

$(1 \# 1) \# 1 \overset{?}{=} 1 \# (1 \# 1) \qquad (1 \$ 1) \$ 1 \overset{?}{=} 1 \$ (1 \$ 1)$

$0 \# 1 \overset{?}{=} 1 \# 0 \qquad 1 \$ 1 = 1 \$ 1$

$1 = 1$ True $\qquad 1 = 1$ True

Since all substitutions result in true statements, the associative property holds for both operations.

c) We can see from the # operation table that 0 is the element in A which is the # identity. Similarly, 1 is the element in A which is the \$ identity and 1 ≠ 0. Hence, each operation has an identity element and these identity elements are not equal.

d) We observe in the # operation table that 0 # 0 = 0 and 1 # 1 = 0 which means that each element in A is its own inverse under the operation of #. Also, 1 \$ 1 = 1 which means that each element, except # identity, 0, has a \$ inverse. Thus, all elements in A have necessary inverses under both operations.

e) Both operations are commutative as may be verified by viewing the tables for # and \$. Note that both tables are symmetrical about the diagonal running from upper-left to lower right.

f) We can check the distribution of \$ over # by making all possible substitutions of 0 and 1 into the expression a \$ (b # c) = (a \$ b) # (a \$ c).

Check for

$$\underline{a \$ (b \# x) = (a \$ b) \# (a \$ c)}$$

$$0 \$ (0 \# 0) \overset{?}{=} (0 \$ 0) \# (0 \$ 0)$$

$$0 \$ 0 \overset{?}{=} 0 \# 0$$

$$0 = 0 \quad \text{True}$$

$$0 \$ (0 \# 1) \overset{?}{=} (0 \$ 0) \# (0 \$ 1)$$

$$0 \$ 1 \overset{?}{=} 0 \# 0$$

$$0 = 0 \quad \text{True}$$

. . .

. . .

$$1 \$ (1 \# 1) \overset{?}{=} (1 \$ 1) \# (1 \$ 1)$$

$$1 \$ 0 \overset{?}{=} 1 \# 1$$

$$0 = 0 \quad \text{True}$$

All substitutions yield true statements and the distribution of $ over # is varified.

Since all of the field properties hold true on the system (A, #, $) it is a field.

In the previous example, you saw that two binary operations defined on a finite set can result in a field. Checking for many of the field properties in such a system is a relatively easy process. The properties of closure and commutativity are easily observed. Inverses and identity elements can be uncovered by careful examination of the operation tables However, the properties of distribution and association are checked only by laborious substitution. A finite set containing twenty elements would require 24,000 substitutions just to check these two properties. This simply could not be done by hand. Since the algorithm for this checking procedure can be completely described and communicated to the computer in BASIC, we will use the computer to check finite mathematical systems for these field properties. This process is demonstrated in the next example.

Example 2-2-5

Given the set $B = \{1, 2, 3\}$, we will use the "three hour clock" shown below to generate two arbitrary binary operations on this set.

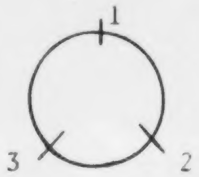


Figure 2-2-6

$\forall a \in B$, $\forall b \in B$, we will define a # b as the number position finally obtained by moving b units clockwise from a in Figure 2-2-6.

We will also define a $ b as the number position finally obtained by moving a units clockwise, from position a, (b - 1) times.

These definitions of # and $ on set $B = \{1, 2, 3\}$ result in the following operation tables.

#	1	2	3
1	2	3	1
2	3	1	2
3	1	2	3

$	1	2	3
1	1	2	3
2	2	1	3
3	3	3	3

Is the system (B, #, $) a field?

a) The system is closed and commutative.

b) The element 3 is the # identity while the element 1, 1 ≠ 3, is the $ identity.

c) The element 1 is the # inverse of the element 2. The element 2 is the # inverse of the element 1. The element 3 is its own # inverse. The elements 1 and 2 are their own $ inverses. The element 3 has no $ inverse. However, if we check field property number 9, inverse property for $, we see that it is not necessary for the # identity to have a $ inverse.

All that remains to show that (B, #, $) is a field is to check the properties of association and distribution. We can do this by using the computer technique described below.

There are two library programs stored in the computer facility available to you. The names of these programs are ASSOC and DISTR. They can be used to perform the following analysis.

ASSOC - Tests the associative property for a binary operation defined on a finite set.

DISTR - Tests the distribution of the binary operation $ over the binary operation # as defined on a finite set.

These programs will check finite operation tables for the associative and distributive field properties. The following constraints apply in using these programs.

1. The number of elements in the set must be less than or equal to 10.
2. The elements of the set must be numbers.
3. The binary operation must be defined by a square table. The elements across the top and down the side are headings and not elements of the table itself. The numbers in these headings must be in the same order from left to right across the top as they are from top to bottom on the left hand side.

Example:

The elements in the headings must be in the same order.

$	1	2	3
1	1	2	3
2	2	1	3
3	3	3	3

4. In each program the set must be closed with respect to the operation if valid results are to be obtained.

The following instructions apply when using ASSOC.

1. Enter the total number of elements in the set as DATA in line 10.

Since $B = \{1, 2, 3\}$

You type:

```
10  DATA 3
```

2. Enter the specific elements of the set as DATA in line 20.

Since $B = \{1, 2, 3\}$

You type:

```
10  DATA 3
20  DATA 1,2,3
```

3. Enter the definition of the binary operation as a square table using lines 30-39 consecutively.

Since $B = \{1, 2, 3\}$ and the operation $ is defined by

$	1	2	3
1	1	2	3
2	2	1	3
3	3	3	3

You type:

```
10  DATA 3
20  DATA 1,2,3
30  DATA 1,2,3
31  DATA 2,1,3
32  DATA 3,3,3
```

The elements in lines 30-32 are the elements of the definition table without the headings.

Be sure to use consecutive lines between 30 and 39. Use as many as you need. If this program is now run on the computer it will check (B, #, $) for the associative property over $. In order to check (B, #, $) for the associative property over #, the # operation table must be entered as DATA in lines 30-39 and the program rerun.

Since the distributive property uses two binary operations defined on the set, the following instructions apply when using DISTR.

1. Delete all program DATA lines 0-99 to remove old DATA. (Procedures for accomplishing this vary according to the computer system in use.)

2. Enter your new DATA as follows.

   a) The number of elements in the set as DATA in line 10.

   b) The specific elements of the set as DATA in line 20.

   c) The definition table for the binary operation ($) as DATA in lines 30-39. Use consecutive line numbers.

   d) The definition table for the binary operation (#) as DATA in lines 40-49. Use consecutive line numbers.

This program checks for distribution of the operation ($) over the operation (#). If the other distribution is to be checked, the DATA in lines 30-39 must be interchanged with the DATA in line 40-49.

Exercise 2-2-7

1. Use the computer to check (B, #, $) from Example 2-2-5 for the associative property on # and $ and the distribution of $ over #.

2. Is (B, #, $) a field?
   Why? Why not?

3. Is the system (C, #) associative with respect to # if:

   C = {+1, -1, + 2, -2}

   and # is defined by

   the table below.

#	2	-2	1	-1
2	-1	1	2	-2
-2	1	-1	-2	-2
1	2	-2	1	-1
-1	-2	2	-1	-1

Exercise 2-2-7 (continued)

4. Is the system (C, #) in Problem 3 commutative with respect to #?

Problem Set 2-2-8

1. Consider the set S, a subset of the real numbers where $S = \{0, 1, 2\}$, and with # and $ defined as follows: (Such a system is called Integers Mod 3.)

#	0,	1,	2
0	0	1	2
1	1	2	0
2	2	0	1

$	0,	1,	2
0	0	0	0
1	0	1	2
2	0	2	1

Analyze the system (S, #, $) to determine if it is a field.

Use the computer wherever applicable.

2. The properties that a system must have in order to be field are listed in the table at the end of this exercise. Investigate each of the systems listed below to determine if it is a field. Unless otherwise indicated, assume the operation # represents addition of real numbers and $ represents multiplication of real numbers.

a) (A, #, $) where $A = \{-1, 0, 1\}$
b) (B, #, $) where $B = \{x | x = 2n - 1, n \in N, x \in I\}$
c) (C, #, $) where $C = \{0, 1, 2, 3, 4\}$

and the operations of # and $ as given below.

#	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	4	1	2
4	4	0	1	2	3

$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

d) (D, #, $) where $D = \{x | x \in Q, x \geq 0\}$
e) (E, #, $) where $E = \{-1, -3, 1, 3\}$
and $\forall a \in E, \forall b \in E, a \# b = a$
$\forall a \in E, \forall b \in E, a \$ b = a/b$

Problem Set 2-2-8 (continued)

Complete the table by placing a "Y" in the square to indicate that the property holds, or an "N" to show that the property does not hold.

Property	(A,#,$)	(B,#,$)	(C,#,$)	(D,#,$)	(E,#,$)
1. Closure #					
2. Associative #					
3. # Identity					
4. # Inverse					
5. Commutative #					
6. Closure $					
7. Associative $					
8. $ Identity					
9. $ Inverse					
10. Commutative $					
11. Distributive $ over #					
Is the system a Field?					

Problem Set 2-2-8 (continued)

3. Given the set L, when L = {1, 3, 7, 9}. Define two binary operations #, $, on L just as you like - except - BE SURE THE SYSTEM (L, #, $) IS A FIELD! You should use the computer wherever appropriate and submit the documentation necessary to prove that your system is indeed a field. Use your creativity to the fullest - and by the way HAVE FUN!

Exercise 2-2-9

A mathematical system denoted (G, !) consisting of a set G and a binary operation ! is called a GROUP if and only if

a) VaεG, VbεG, a ! b ε G
b) VaεG, there exists e ε G such that a ! e = a
c) VaεG, there exists a' ε G such that a ! a' = e
d) VaεG, VbεG, VcεG, (a ! b) ! c = a ! (b ! c)

1. Is (G, !) a group when G = {0, 1, 2} and ! is defined by the table below?

!	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

2. Is (G, !) a group when G = {0, 1, -1} and ! is addition as normally defined?

3. Is (G, !) a group when G = {1, 3, 7, 9} and ! is defined by the table below?

!	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

*2-3 Axioms For The Algebra Of The Real Numbers

In the previous work we developed the system of real numbers. This system, consisting of the set of real numbers, and the binary operations of addition and multiplication, was shown to have all the properties of a field.

We will now use the eleven field properties along with some additional definitions and principles of logic to demonstrate the development of the mathematical structure known as the algebra of the real numbers.

We will accept the following eleven field properties of the set of real numbers R as the axioms for the algebra of the real numbers.

Axiom 2-3-1 For each a and $b \epsilon R$, there is a unique sum $(a + b) \epsilon R$. (ClA)

Axiom 2-3-2 For each a, b and $c \epsilon R$, $(a + b) + c = a + (b + c)$. (APA)

Axiom 2-3-3 There exists a real number $0 \epsilon R$, such that for each $a \epsilon R$, $a + 0 = 0 + a = a$. (AId)

Axiom 2-3-4 For each $a \epsilon R$, there exists a real number, denoted by $-a \epsilon R$, such that $-a + a = a + -a = 0$. (AIn)

Axiom 2-3-5 For each a and $b \epsilon R$, $a + b = b + a$. (CPA)

Axiom 2-3-6 For each a and $b \epsilon R$ there is a unique product $a \cdot b \epsilon R$. (ClM)

Axiom 2-3-7 For each a, b and $c \epsilon R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (APM)

Axiom 2-3-8 There exists a real number $1 \epsilon R$, $1 \neq 0$, such that for each $a \epsilon R$, $a \cdot 1 = 1 \cdot a = a$. (MId)

Axiom 2-3-9 For each $a \epsilon R$, $a \neq 0$, there exists an element $\frac{1}{a} \epsilon R$ such that $\frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1$. (MIn)

Axiom 2-3-10 For each a and $b \epsilon R$, $a \cdot b = b \cdot a$. (CPM)

Axiom 2-3-11 For each a, b and $c \epsilon R$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$. (DMA)

Most of the variables throughout the remainder of this text will have the set of real numbers as their replacement set. Hence, we redefine the notation Va to mean $\text{V}a \epsilon R$. The replacement set for a variable will be stated only when it is a set other than R. For example, if the replacement set is N then we will write $\text{V}a \epsilon N$. If a replacement set is not stated then it is assumed to be R. In addition the symbol $\exists a$ will be used in place of the phrase: "There exists $a \epsilon R$." Using these abbreviations, Axiom 2-3-9 would be written as follows.

$$\forall a \neq 0,\ \exists\ 1/a \text{ such that } \frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1$$

Exercise 2-3-12

Rewrite axioms 2-3-1, 2-3-2, 2-3-3 using the abbreviations ∀ and ∃.

*2-4 Principles of Logic

In this section we will introduce notation and state the principles of logic which are necessary for a deductive development of the algebra of the real numbers

Theorems are often stated in the form "if a then b". This type of sentence is called a conditional sentence or an implication. We will use the symbol "→" to replace the words "if-then." The implication "If a then b" would be written $a \to b$. In the sentence $a \to b$, a is called the antecedent and b is called the consequent.

Example 2-4-1

The theorem ∀x, If $x = 3$ then $x^2 = 9$ would be written

$$\forall x,\ x = 3 \to x^2 = 9$$

In many logical arguments, a reasoning pattern with the following form is used.

$$\frac{p,\ p \to q}{q}$$

The letters p and q represent sentences. The sentences above the horizontal line are called the hypotheses. The sentence below the line is called the conclusion. This reasoning pattern always gives a valid conclusion from the given hypotheses. The pattern is called modus ponens.

Example 2-4-2

$$\frac{p,\ p \to q}{q}$$

Your father is president.
If your father is president then he is over forty.
He is over forty.

In this example the letter p represents the sentence, "Your father is president." The sentence, "He is over forty." is represented by the letter q. This conclusion is valid by the modus ponens reasoning pattern.

Exercise 2-4-3

In each of Exercises 1 - 4, write the conclusion (3) which follows from the given hypotheses (1) and (2) according to the rule of modus ponens. You may read the horizontal bar as "therefore".

1. 1) $a = b$

2) $\underline{(a = b) \rightarrow (b = a)}$

3)

2. 1) $a = b$

2) $\underline{(a = b) \rightarrow (a + c = b + c)}$

3)

3. 1) $cd = 0$

2) $\underline{(cd = 0) \rightarrow (c = 0 \text{ or } d = 0)}$

3)

4. 1) $b = -5$

2) $\underline{(b = -5) \rightarrow (b^2 = +25)}$

3)

In each of Exercises 5 - 8 supply the missing hypothsis so that sentence (3) follows from sentences (1) and (2) by the rule of modus ponens.

5. 1) $a = -8$

2) ____________

3) $c + e = d + e$

6. 1) $a = -b$

2) ____________

3) $a^2 = -b^2$

7. 1)

2) $\underline{(+7 = +2) \rightarrow (+2 = +7)}$

3) $+2 = +7$

8. 1)

2) $\underline{(t + f = 0) \rightarrow (f = -t)}$

3) $f = -t$

9. Here are two test-patterns. In one of them conclusion (3) follows from the hypotheses (1) and (2) by the rule of modus ponens. In the other conclusion (3) does not follow logically from hypotheses (1) and (2). Which test-pattern shows the use of modus ponens?

a) 1) $a = -8$

2) $\underline{(a = -8) \rightarrow (a^2 = +64)}$

3) $a^2 = +64$

b) 1) $a^2 = +64$

2) $\underline{(a = -8) \rightarrow (a^2 = +64)}$

3) $a = -8$

Another principle of logic which we will use in the development of the algebra of the real numbers is called the substitution rule. The substitution rule states that given an equation and a second sentence, either side of the equation may be used to replace the other side anywhere in the second sentence.

Example 2-4-4

From the equation $x = 3$ and the sentence $2x = y$, we can derive the sentence $2(3) = y$ by the substitution rule.

Exercise 2-4-5

Given sentences 1 and 2, form a third sentence using the substitution rule.

a) 1) $a = b + c$
   2) $c = 2a$

b) 1) $3(x + y)$ is even.
   2) $x = 9$

c) 1) $x = z^3$
   2) $y = 3x^2$

d) 1) $c + d = -2$
   2) $a = b - (c + d)$

e) 1) $x = -4$
   2) $y = -x^2$

f) 1) $a + c = 9$
   2) $-9 = y$

Another logical principle is the Reflexive Property of Equality (RPE). Application of this principle to the set of real numbers results in the following generalization.

$$\forall x,\ x = x$$

We will accept this generalization as an axiom. Two other Principles of Equality will be proved later.

We often want to derive conditional sentences of the form $p \rightarrow q$ (if p then q). We sometimes begin such a derivation by stating p as an assumption or hypothesis. Then we use a series of deductive steps to derive q. This series of steps is completed by stating the conditional "If p then q". The rule of logic which justifies this conditional sentence is called the Deduction Rule. The Deduction Rule is sometimes referred to as conditionalizing a conclusion and discharging an assumption.

Example 2-4-6

Prove that $\forall a, \forall b, \forall c \quad a = b \rightarrow a + c = b + c$

1.	$a = b$	1.	Assumption
2.	$a + c = a + c$	2.	RPE
3.	$a + c = b + c$	3.	(1) (2) Substitution Rule
4.	If $a = b$ then $a + c = b + c$	4.	(1) (3) Deduction Rule

In step (4) the deduction rule was used to form a conditional sentence in which the conclusion, $a + c = b + c$, was conditionalized and the assumption, $a = b$, was discharged.

This deduction rule allows us to finish the proof with a statement of the conditional sentence which we set out to prove. It emphasizes the fact that the conclusion, $a + c = b + c$, is dependent upon the assumption, $a = b$.

Exercise 2-4-7

Write a valid proof for the generalization $\forall a, \forall b, \forall c, (a = b) \rightarrow (c + a = c + b)$. Be sure to include all necessary principles of logic.

Combining the two generalizations proved in Example 2-4-6 and Exercise 2-4-7, we obtain the following generalized statement.

$$\forall a, \forall b, \forall c, \ a = b \rightarrow a + c = b + c \text{ and}$$

$$a = b \rightarrow c + a = c + b$$

This, we will denote as the Addition Property of Equality (APE).

Exercise 2-4-8

Prove the Multiplication Property of Equality (MPE)

$\forall a, \forall b, \forall c, \ a = b \rightarrow a \cdot c = b \cdot c$ and $a = b \rightarrow c \cdot a = c \cdot b$.

Be sure to include all necessary principles of logic.

Example 2-4-9 Symmetric Property of Equality (SPE)

Prove $\forall a, \forall b \ \ a = b \rightarrow b = a$

Proof:

1. $a = b$	1. Assumption
2. $a = a$	2. RPE
3. $b = a$	3. (1) (2) Substitution Rule
4. If $a = b$ then $b = a$	4. Deduction Rule

Example 2-4-10 Transitive Property of Equality (TPE)

Prove $\forall a, \forall b, \forall c, a = b \text{ and } b = c \rightarrow a = c$

Proof:

1.	$a = b$	1.	Assumption
2.	$b = c$	2.	Assumption
3.	$a = c$	3.	(1) (2) Substitution Rule
4.	If $a = b$ and $b = c$ then $a = c$	4.	Deduction Rule

We have discussed sentences of the form $p \rightarrow q$. These were called conditional sentences and this symbolism represents "If p then q". Often we wish to consider sentences of the form $p \leftrightarrow q$. Such sentences are called biconditional sentences. The symbolism $p \leftrightarrow q$ is read "p if and only if q". The sentence "p if and only if q" means if p then q and if q then p. In order to produce a valid proof of such a biconditional sentence both of the conditionals $p \rightarrow q$ and $q \rightarrow p$ must be proved.

Example 2-4-11 The Biconditional Sentence

The sentence,

$$x^2 = 9 \text{ if and only if } x = 3 \text{ or } x = -3$$

is represented by,

$$x^2 = 9 \leftrightarrow (x = 3 \text{ or } x = -3)$$

which means the same as the conjunction of the two conditional sentences,

$$x^2 = 9 \rightarrow (x = 3 \text{ or } x = -3)$$

and $$(x = 3 \text{ or } x = -3) \rightarrow x^2 = 9.$$

Summary of Properties of Equality.

Reflexive Property of Equality (RPE)

$\forall x, x = x$

Symmetric Property of Equality (SPE)

$\forall a, \forall b, \quad a = b \rightarrow b = a$

Transitive Property of Equality (TPE)

$\forall a, \forall b, \forall c, \quad a = b \text{ and } b = c \rightarrow a = c$

Addition Property of Equality

$$\forall a, \forall b, \forall c, \quad a = b \rightarrow a + c = b + c \text{ and } a = b \rightarrow c + a = c + b$$

Multiplication Property of Equality

$$\forall a, \forall b, \forall c, \quad a = b \rightarrow a \cdot c = b \cdot c \text{ and } a = b \rightarrow c \cdot a = c \cdot b$$

*2-5 Development of the Algebraic Structure

In this section we will prove algebraic theorems which are dependent upon our stated axioms and can be derived using the logical principles from Section 2-4. This will demonstrate how the theorems of algebra can be proved deductively.

Theorem 2-5-1 Cancellation Principle for Addition (Can PA)

$\forall a, \forall b, \forall c$ If $a + c = b + c$ then $a = b$ and $\forall a, \forall b, \forall c$ If $c + a = c + b$ then $a = b$.

Proof of $a + c = b + c \rightarrow a = b$

1. $a + c = b + c$	1. Assumption
2. If $a + c = b + c$ then $(a + c) + -c = (b + c) + -c$	2. APE
3. $(a + c) + -c = (b + c) + -c$	3. (1) (2) Modus Ponens
4. $(a + c) + -c = a + (c + -c)$	4. APA
5. $(b + c) + -c = b + (c + -c)$	5. APA
6. $a + (c + -c) = b + (c + -c)$	6. (3) (4) (5) Sub. Rule
7. $c + -c = 0$	7. AIn
8. $a + 0 = b + 0$	8. (6) (7) Sub. Rule
9. $a + 0 = a$ and $b + 0 = b$	9. AId
10. $a = b$	10. (8) (9) Sub. Rule
11. $a + c = b + c \rightarrow a = b$	11. (1) (10) Deduction Rule

Exercise 2-5-2

1. Prove that
$$\forall a, \forall b, \forall c, (c + a = c + b) \rightarrow (a = b)$$

2. Does Theorem 2-5-1 combine with the generalization in Example 2-4-6 to form a biconditional sentence?
If so, state the biconditional sentence.
If not, why not?

Theorem 2-5-3 Cancellation Principle for Multiplication (Can PM)

$\forall a, \forall b, \forall c, (a \cdot c = b \cdot c \text{ and } c \neq 0) \rightarrow a = b$ and $(c \cdot a = c \cdot b, \text{ and } c \neq 0) \rightarrow a = b$

1.	$a \cdot c = b \cdot c$	1. Assumption
2.	$c \neq 0$	2. Assumption
3.	If $c \neq 0$ then $\exists \frac{1}{c} \in R$ such that $c \cdot \frac{1}{c} = 1$	3. MIn
4.	$\exists (1/c) \in R$ such that $c \cdot (1/c) = 1$	4. (2) (3) Modus Ponens
5.	If $a \cdot c = b \cdot c$ then $(a \cdot c) \cdot 1/c = (b \cdot c) \cdot 1/c$	5. MPE
6.	$(a \cdot c) \cdot 1/c = (b \cdot c) \cdot (1/c)$	6. (1) (5) Modus Ponens
7.	$(a \cdot c) \cdot 1/c = a \cdot (c \cdot (1/c))$	7. APM
8.	$(b \cdot c) \cdot 1/c = b \cdot (c \cdot (1/c))$	8. APM
9.	$a \cdot (c \cdot 1/c) = b \cdot (c \cdot (1/c))$	9. (6)(7)(8) Substitution Rule
10.	$a \cdot 1 = b \cdot 1$	10. (4) (9) Sub. Rule
11.	$a \cdot 1 = a$	11. MId
12.	$b \cdot 1 = b$	12. MId
13.	$a = b$	13. (10) (11) (12) Sub. Rule
14.	$(a \cdot c = b \cdot c \text{ and } c \neq 0) \rightarrow a = b$	14. Deduction Rule

Exercise 2-5-4

Prove that
$\forall a, \forall b, \forall c, (c \cdot a = c \cdot b \text{ and } c \neq 0) \rightarrow a = b$

Theorem 2-5-5 Uniqueness of Zero

$\forall a, \forall b \quad a + b = a \rightarrow b = 0$

1. $a + b = a$	1. Assumption
2. $a + 0 = a$	2. AId
3. $a + b = a + 0$	3. (1) (2) Sub. Rule
4. $b = 0$	4. Can PA
5. $a + b = a \rightarrow b = 0$	5. Deduction Rule

Theorem 2-5-6 Uniqueness of Each Additive Inverse

$\forall a, \forall b \quad a + b = 0 \rightarrow b = -a$

1. $a + b = 0$	1. Assumption
2. $a + -a = 0$	2. AIn
3. $a + b = a + -a$	3. (1) (2) Substitution Rule
4. $b = -a$	4. Can PA
5. $a + b = 0 \rightarrow b = -a$	5. Deduction Rule

Theorem 2-5-7 Principle for Multiplication by Zero

$\forall a \quad a \cdot 0 = 0 \cdot a = 0$

1. $a(a + 0) = a \cdot a + a \cdot 0$	1. DMA
2. $a + 0 = a$	2. AId
3. $a \cdot a = a \cdot a + a \cdot 0$	3. (1) (2) Substitution Rule
4. $a \cdot a = a \cdot a + a \cdot 0 \rightarrow a \cdot a + a \cdot 0 = a \cdot a$	4. SPE

5. $a \cdot a + a \cdot 0 = a \cdot a$	5. (3) (4) Modus Ponens
6. $a \cdot a + a \cdot 0 = a \cdot a \rightarrow a \cdot 0 = 0$	6. Theorem 2-5-5 Uniqueness of Zero
7. $a \cdot 0 = 0$	7. (5) (6) Modus Ponens
8. $a \cdot 0 = 0 \cdot a$	8. CPM
9. $0 \cdot a = 0$	9. (7) (8) Substitution Rule
10. $a \cdot 0 = 0 \cdot a = 0$	10. (7) (8) (9) Sub. Rule

Theorem 2-5-8

$\forall a, \forall b \quad a(-b) = -(ab) = (-a)b$

Proof:

1. $ab + a(-b) = a(b + -b)$	1. DMA
2. $b + -b = 0$	2. AIn
3. $ab + a(-b) = a \cdot 0$	3. (1) (2) Substitution Rule
4. $a \cdot 0 = 0$	4. Theorem 2-5-7 Mult. by Zero
5. $ab + a(-b) = 0$	5. (3) (4) Substitution Rule
6. $ab + a(-b) = 0 \rightarrow a(-b) = -(ab)$	6. Theorem 2-5-6
7. $a(-b) = -(ab)$	7. (5) (6) Modus Ponens

Exercise 2-5-9

Prove: $\forall a, \forall b, -(ab) = (-a)b$

Note that this generalization is not the definition of multiplication of signed numbers. The generalization is true for any real number substitutions for a and b, positive or negative.

Theorem 2-5-10

$\forall a \quad a = -(-a)$

Theorem 2-5-10 (continued)

1. $a + -a = 0$	1. AIn
2. $a + -a = -a + a$	2. CPA
3. $-a + a = 0$	3. (1) (2) Substitution Rule
4. If $-a + a = 0$ then $a = -(-a)$	4. Theorem 2-5-6
5. $a = -(-a)$	5. (3) (4) Modus Ponens

Theorem 2-5-11

$\forall a, \forall b \quad (-a)(-b) = ab$

1. $(-a)(-b) = -((-a)b)$	1. Theorem 2-5-8
2. $((-a) \cdot b) = -(ab)$	2. Theorem 2-5-8
3. $(-a)(-b) = --(ab)$	3. (1) (2) Substitution Rule
4. $ab = --(ab)$	4. Theorem 2-5-10
5. $(-a)(-b) = ab$	5. (3) (4) Substitution Rule

Exercise 2-5-12

1. Prove $\forall a, \forall b, \quad a \neq 0$ and $ab = a \rightarrow b = 1$

2. Prove $\forall a, \forall b, \quad -(a + b) = -a + -b$

3. Prove $\forall a, \ -1 \cdot a = -a$

You may have noticed in our development of the algebraic structure that we have only dealt with theorems involving the operations of addition and multiplication. We will now expand this structure to include the definitions of subtraction and division, proving several theorems related to these operations.

Definition 2-5-13 Subtraction of Real Numbers

$$\forall a, \forall b \quad a - b = a + -b$$

This definition states that to subtract a real number you may add its opposite.

Theorem 2-5-14 Removing Symbols of Grouping Preceded by a Minus Sign

$$Va, Vb, Vc \quad a - (b + c) = a - b - c$$

Proof:

1.	$a - (b + c) = a + -(b + c)$	1. Definition 2-5-13
2.	$-(b + c) = -b + -c$	2. Exercise 2-5-12 no. 12
3.	$a - (b + c) = a + (-b + -c)$	3. (1) (2) Substitution Rule
4.	$a + (-b + -c) = (a + -b) + -c$	4. APA
5.	$a - (b + c) = (a + -b) + -c$	5. (3) (4) Substitution Rule
6.	$a + -b = a - b$	6. Definition 2-5-13
7.	$a - (b + c) = (a - b) + -c$	7. (5) (6) Substitution Rule
8.	$(a - b) + -c = (a - b) - c$	8. Definition 2-5-13
9.	$a - (b + c) = (a - b) - c$	9. (7) (8) Substitution Rule
10.	$(a - b) - c = a - b - c$	10. Convention for Omitting Group Symbols
11.	$a - (b + a) = a - b - c$	11. (9) (10) Substitution Rule

Exercise 2-5-15

1. Prove $Va, Vb, Vc \quad a - (b - c) = a - b + c$
2. Prove $Va, Vb, Vc \quad a - b = c \leftrightarrow a = c + b$

Definition 2-5-16 Division of Real Numbers

$$Va, Vb \neq 0 \quad a \div b = a/b = a \cdot \frac{1}{b}$$

Theorem 2-5-17

$$Va, Vb \neq 0, Vc, cb = a \leftrightarrow c = \frac{a}{b}$$

Theorem 2-5-17 (continued)

Proof:

1.	$cb = a$	1. Assumption
2.	$b \neq 0$	2. Assumption
3.	$\exists\ (1/b) \in R$ such that $b \cdot (1/b) = (1/b) \cdot b = 1$	3. MIn
4.	If $cb = a$ then $cb \cdot (1/b) = a \cdot (1/b)$	4. MPE
5.	$cb \cdot (1/b) = a \cdot (1/b)$	5. (1) (4) Modus Ponens
6.	$cb \cdot (1/b) = c \cdot (b \cdot (1/b))$	6. APM
7.	$c \cdot (b \cdot (1/b)) = a \cdot 1/b$	7. (5) (6) Substitution Rule
8.	$b \cdot (1/b) = 1$	8. MIn
9.	$c \cdot 1 = a \cdot (1/b)$	9. (7) (8) Substitution Rule
10.	$c \cdot 1 = c$	10. MId
11.	$c = a \cdot (1/b)$	11. (9) (10) Substitution Rule
12.	If $cb = a$ and $b \neq 0$ then $c = a \cdot (1/b) = a/b$	12. Deduction Rule

Exercise 2-5-18

The proof of theorem 2-5-17, above, is not complete. This is due to the biconditional nature of the generalization. Complete the proof of this theorem:

$$\forall a,\ \forall b,\ b \neq 0,\ \forall c \quad cb = a \leftrightarrow c = \frac{a}{b}$$

To this point we have seen how the structure of algebra can be developed deductively from the field $(R, +, \cdot)$ and a few principles of logic.

A review of all properties and theorems proved in this section is given below.

APE Addition Property of Equality

$\forall a, \forall b, \forall c \quad a = b \rightarrow a + c = b + c$ and $a = b \rightarrow c + a = c + b$

MPE Multiplication Property of Equality

$\forall a, \forall b, \forall c \quad a = b \rightarrow ac = bc$ and $a = b \rightarrow ca = cb$

SPE Symmetric Property of Equality

$\forall a, \forall b \quad a = b \rightarrow b = a$

TPE Transitive Property of Equality

$\forall a, \forall b, \forall c \quad a = b \text{ and } b = c \rightarrow a = c$

Can PA Cancellation Principle for Addition

$\forall a, \forall b, \forall c \quad a + c = b + c \rightarrow a + b \text{ and } c + a = c + b \rightarrow a = b$

Can PM Cancellation Principle for Multiplication

$\forall a, \forall b, \forall c \quad (ac = bc \text{ and } c \neq 0) \rightarrow a = b \text{ and } (ca = cb \text{ and } c \neq 0) \rightarrow a = b$

Theorem 2-5-5 Uniqueness of Zero

$\forall a, \forall b \quad a + b = a \rightarrow b = 0$

Theorem 2-5-6 Uniqueness of Each Additive Inverse

$\forall a, \forall b \quad a + b = 0 \rightarrow b = -a$

Theorem 2-5-7 Principle for Multiplication by Zero

$\forall a \quad a \cdot 0 = 0 \cdot a = 0$

Theorem 2-5-8

$\forall a, \forall b \quad a(-b) = -(ab) = (-a)b$

Theorem 2-5-9

$\forall a \quad a = -(-a)$

Theorem 2-5-10

$\forall a, \forall b \quad (-a)(-b) = ab$

Theorem 2-5-11

$\forall a, \forall b \quad (a \neq 0 \text{ and } ab = a) \rightarrow b = 1$

Theorem 2-5-12

$\forall a, \forall b \quad -(a + b) = -a + -b$

Theorem 2-5-13

$\forall a \quad -1 \cdot a = -a$

Theorem 2-5-14

$$\forall a, \forall b, \forall c \quad a - (b + c) = a - b - c$$

Theorem 2-5-15

$$\forall a, \forall b, \forall c \quad a - (b - c) = a - b + c$$

Theorem 2-5-16

$$\forall a, \forall b, \forall c \quad a - b = c \leftrightarrow a = c + b$$

Theorem 2-5-17

$$\forall a, \forall b \neq 0, \forall c \quad cb = a \leftrightarrow c = a/b$$

There are many other theorems that could be derived from the axioms and definitions which we have stated. The structure of the algebra of real numbers could be completely developed deductively from these axioms and definitions. Since this is a time consuming task we do not propose such a development in this text. You may wish to show how some of the following theorems might be proved, but we will state them here without proof. Applications of these theorems make it possible to transform and simplify algebraic expressions and to solve equations.

Theorem 2-5-18

$$\forall a, \forall b \quad ab = 0 \leftrightarrow (a = 0 \text{ or } b = 0)$$

Theorem 2-5-19

$$\forall a \neq 0 \quad \frac{1}{\frac{1}{a}} = a$$

Theorem 2-5-20

$$\forall a, \forall b \neq 0 \quad \frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}$$

Theorem 2-5-21

$$\forall a, \forall b \neq 0 \quad \frac{-a}{-b} = \frac{a}{b}$$

Theorem 2-5-22

$$\forall a, \forall b \neq 0, \forall c \quad \frac{a}{b} \cdot c = \frac{ac}{b} = c \cdot \frac{a}{b}$$

Theorem 2-5-23

$$Va,\ Vb \neq 0,\ Vc \neq 0 \quad \frac{a}{\frac{b}{c}} = a \cdot \frac{c}{b}$$

Theorem 2-5-24

$$Va,\ Vb,\ Vc \neq 0 \quad \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

Theorem 2-5-25

$$Va,\ Vb,\ Vc \neq 0 \quad \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$$

Theorem 2-5-26

$$Va \neq 0,\ Vb,\ Vc \quad \frac{ab + ac}{a} = b + c$$

Theorem 2-5-27

$$Va,\ Vb \neq 0,\ Vc,\ Vd \neq 0 \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Theorem 2-5-28

$$Va,\ Vb \neq 0,\ Vc,\ Vd \neq 0 \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Theorem 2-5-29

$$Va,\ Vb \neq 0,\ Vc,\ Vd \neq 0 \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

Theorem 2-5-30

$$Va,\ Vb \neq 0,\ Vc \neq 0 \quad \frac{a}{b} = \frac{ac}{bc}$$

Theorem 2-5-31

$$Va,\ Vb,\ Vc \quad a(b - c) = ab - ac$$

Theorem 2-5-32

$$Va,\ Vb,\ Vc \quad (a - b)c = ac - bc$$

Exercise 2-5-33

Each of the following sentences is an instance of one of the theorems or properties listed above. Identify the one theorem or property which justifies each of the following as a true statement.

1. $7(x - y) + (2z - 4) = 7(x - y) \rightarrow (2z - 4) = 0$
2. $(x + y - z) + (-x - y + z) = 0 \rightarrow (-x - y + z) = -(x + y - z)$
3. $x + 3 = 7$ and $7 = x - 3 + 6 \rightarrow x + 3 = x - 3 + 6$
4. $-x + y - z = -(-[-x + y - z])$
5. $7x - 2 + 3y = 7x - 2 + 3 \rightarrow 3y = 3$
6. If $x(y + 3) = 7(y + 3)$ and $y + 3 \neq 0$ then $x = 7$
7. $2(x - 7) + 3 = 5x + 3 \rightarrow 2(x - 7) = 5x$
8. $2x + 3y = 7 \rightarrow x(2x + 3y) = x \cdot 7$
9. $[-(x - y)][-(xz)] = (x - y)(xz)$
10. $[a(b - c)] \cdot 0 = 0$
11. $(-x)(2x + 3) = 7 \rightarrow 7 = (-x)(2x + 3)$
12. $(a + 3b)x = 29 - c \rightarrow (a + 3b)x + c = 29 - c + c$
13. $(3x - 13)y = 3x - 13 \rightarrow y = 1$
14. $\frac{3x + 2}{29} = \frac{3x}{29} + \frac{2}{29}$
15. $\frac{(x^2 - 49)}{x + 3} = \frac{(x^2 - 49)}{x + 3} \cdot \frac{(x - 8)}{(x - 8)}$
16. $3a - 12 + (2n + 7) = 3a - (12 - (2n + 7))$
17. $(27b - 13c) - (25x + z) = m \rightarrow 27b - 13c = m + 25x + z$
18. $(\frac{13x}{4} + \frac{17y}{3}) = -\frac{13x}{4} + -\frac{17y}{3}$
19. $\frac{-2}{3} = \frac{2}{-3}$

20. $\frac{(x - 3)}{5} - \frac{(7y + 2)}{4} = \frac{(x - 3) \cdot 4 - 5(7y + 2)}{5 \cdot 4}$

21. $(x - 2)(x + 3) = 0 \rightarrow x - 2 = 0$ or $x + 3 = 0$

22. $\frac{(x^2 + x - 2)(3)}{(x - 2)} = (3)\frac{(x^2 + x - 2)}{(x - 2)}$

23. $(m - n)(s + 3) = 49 \rightarrow m - n = \frac{49}{s + 3} \quad s + 3 \neq 0$

24. $\dfrac{\frac{1}{3}}{\frac{x + 2}{3}} = \frac{1}{3} \cdot \frac{3}{x + 2}$

25. $\dfrac{1}{\frac{1}{x - 7}} = x - 7$

The following expressions are to be used in answering Exercises 26 through 29. Assume $a, b \in R$. Each expression may be used once, more than once, or not at all.

a) $a - b$

b) $b + a$

c) $(-a)(-b)$

d) $-(-a)(-b)$

e) $-a - b$

f) $b - a$

g) $(a)(-b)$

h) $-(a)(-b)$

i) $-(a - b)$

j) $-(b - a)$

k) $(-a)(b)$

l) $-a(-a)(b)$

m) $-(a + b)$

n) $-(-a - b)$

o) ba

p) $-ab$

26. a) Which expressions represent the same real number as $a + b$?
    b) Which expressions represent the additive inverse of $a + b$?

27. a) Which expressions represent the same real number as $a - b$?
    b) Which expressions represent the additive inverse of $a - b$?

28. Which expressions represent the same real number as ab?

29. Which expressions represent the additive inverse of ab?

*2-6 Miscellaneous Review Problems

Review Problem Set 2-6-1

Perform the indicated operations and simplify.

1. 3^2
2. 4^3
3. 6^3
4. $3^2 \cdot 2^3$
5. $10^5 \cdot 10^8$
6. $(2^3 \cdot 3^2)^5$
7. $(2^6 \cdot 7^2)^3$
8. $(2a^2)^2(3^2a)$
9. $(-2ab)^2(a^2)^3$
10. $(-2ab^2)^5(-5a^3)^4$
11. $(3b^{2b})(b^n)^k$, $n, k \in N$
12. $(9a^2)^n \cdot a^{2b}$, $n, b \in N$
13. $\frac{2^7}{2^2}$
14. $\frac{5^8}{5^3}$
15. $\frac{a^3}{a^2}$
16. $\frac{3^4 \cdot 10^2}{3^2 \cdot 10^7}$
17. $\frac{2^3x^9}{6^3x^2}$
18. $\frac{a^7b^3c^9x}{a^5b^5cx^7}$
19. $\frac{(-2x^3x^2)^5}{(-2x^2)^3}$
20. $\frac{(a + 7)^5}{(a + 7)^2}$
21. $(\frac{(x - 1)^3}{(x - 1)})^5$
22. $\frac{(x + 2)^2(x - 1)^3}{(x - 1)^2(x + 2)}$

Review Problem Set 2-6-2

Expand each of the following indicated products

1. $(2x - y)(a + b + c)$
2. $(x + 3)(x - 3)$
3. $(x + 4)^2$
4. $(x - 7)^2$
5. $(x + 5)(x + 7)$
6. $(x - 5)(x + 7)$
7. $(x - 7)(x + 5)$
8. $(x - 7)(x - 5)$
9. $(2x - 3)(2x + 3)$
10. $(2x - 5)(3x - 7)$

Review Problem Set 2-6-3

Factor the following expressions completely.

1. $6p - 3q + 15r$
2. $10y - 5x + 20n - 10z$
3. $a(x + y) + b(x + y)$
4. $3(a - b) - (a - b)$
5. $bx - by - cx + cy$
6. $4m^2 - 9n^2$
7. $y^2 + 16y + 64$
8. $7c^4 - 63$
9. $(x - y + 1)^2 - 1$
10. $x^2 + 3x + 5x + 15$
11. $w^2 -11w + 24$
12. $3a^2 - 4a - 15$
13. $4x^2 - 5x - 6$
14. $y^2 - 10y + 25$
15. $x^4 - 9x^2$
16. $-72k^2 + 2$
17. $8a^2 + 22a^2x - 6a^2x^2$
18. $3x^{2n} - x^n - 2x^{3n}$
19. $4(a + b)^2 + 12(a + b) + 9$
20. $a^2 + b^2 + 2ab - 2a - 2b + 1$

Review Problem Set 2-6-4

Write without parenthesis, simplify.

1. $6(x + y) + x(4 + y) + 3x$

Review Problem Set 2-6-4 (continued)

2. $-3x(-5x^2 + 2 - y)$

3. $3x - (x + 3) - (-5 + 2x - 4)$

4. $a(a - 3) + 4a(a - 3)$

Factor

5. $r(a - b) - s(a - b)$

6. $xa - xb - y(a - b)$

7. $mx - my - ny + nx$

8. $(\sqrt{2} - 3) + 2(3 - \sqrt{2})$

Perform indicated operations and simplify.

9. $\frac{27a^4b^3c^2}{x^2m^2n} \cdot \frac{x^2y^2}{b^2c^2} \cdot \frac{m^3n}{-3a^4y^2}$

10. $\frac{(x - y)^2}{a + b} \cdot \frac{(a + b)^2}{(x - y)^3}$

11. $\frac{4y^2}{\frac{y}{x - y}}$

12. $\frac{\frac{9a^2b^2}{4xy^3}}{\frac{3ab}{2x^3y}}$

13. $\frac{3a}{5} - \frac{2}{5a}$

14. $\frac{2x}{5y} \cdot \frac{15y^3}{20}$

15. $\frac{2a - b}{3a} + \frac{2a - 3b}{2b}$

16. $\frac{x + y}{3x - 9y} \cdot \frac{12}{x + y}$

17. $\frac{x + 5}{x - 5} - \frac{x - 5}{x + 5}$

Review Problem Set 2-6-4 (continued)

18. $\frac{3c}{4bc - bc} \cdot \frac{2b^2 - 4b^2}{12b}$

19. $\frac{1}{m + 2} \cdot \frac{1}{2 - m}$

20. $\frac{\frac{x^2 - xy}{4}}{x^2 - 2xy + y^2}$

21. $\frac{m - n}{m^2 - 1} \cdot \frac{m - 1}{m^2 - n^2}$

22. $\frac{x - 1}{2x^2 - 18} - \frac{x + 2}{3x^2 + 9x}$

23. $\frac{a^2 - 2a - 15}{a^2 - 9} \cdot \frac{a^2 - 6a + 9}{3 - a}$

24. $\frac{x^3}{y^3} \div \frac{x^2 + xy}{xy - y^2}$

25. $\frac{1}{2x^2 + 7x - 15} - \frac{1}{x^2 + 6x + 5}$

26. $\frac{\frac{4}{x} - x}{1 + \frac{2}{x}}$

27. $a - b - \frac{a^2 + b^2}{a + b}$

28. $\frac{m^2 + 6m + 9}{m - 3} - (m - 3)$

29. $x - \frac{x^2 + 3xy}{x - 3} + 3y$

30. $\frac{\frac{a}{1 - a} + \frac{1 + a}{a}}{\frac{1 - a}{a} + \frac{a}{1 + a}}$

Chapter 3

Equations and Inequations.

3-1 Introduction

In this chapter a complete process for solving equations and inequations will be presented. You will find many familiar definitions and theorems as well as some which will be new to you. These definitions and theorems along with the real number properties will provide the means for solving equations and inequations.

Many of the definitions and theorems in the chapter will refer to both equations and inequations while others will only refer to one of these types of sentences. Observe carefully, which relation is involved in each definition.

3-2 Equivalent Expressions

Definition 3-2-1 Equivalent Expressions

Two expressions are equivalent if and only if they name the same number or they have the same value for any numerical replacements for the variables which occur in them.

Here are some equivalent expressions:

$7 + 3$ and $14 - 4$

$3(x + 1)$ and $3x + 3$

$7x + 4 + 2x$ and $9x + 4$

$a - 3b$ and $-(3b - a)$

$7x - 3y$ and $(7x + 4) - (4 + 3y)$

Do you see how to show that these expressions have the same value for each numerical replacement for variables which occur in them?

Another example of expressions which are equivalent may be seen below.

Example 3-2-2

Consider the expression $dfx^2 + (dg + ef)x + eg$, $d,e,f,g,x \in R$
Since, $(dg + ef)x = dgx + efx$ substitution shows that

$\forall d,e,f,g,x \in R$ $dfx^2 + (dg + ef)x + eg$ and $dfx^2 + dgx + efx + eg$

are equivalent expressions as are

$dfx^2 + dgx + efx + eg$ and $dx(fx + g) + e(fx + g)$

and $(dx + e)(fx + g)$.

The final form is a factored form of the expression,

$dfx^2 + (dg + ef)x + eg$.

As an illustration: $(d = 2,\ e = -5,\ f = 3,\ g = 1)$

$$\begin{aligned}\forall x\ 6x^2 + -13x + -5 &= 2 \cdot 3x^2 + (2(1) + (-5)3)x + (-5)(1)\\ &= 2(3)x^2 + 2(1)x + (-5)(3)x + (-5)(1)\\ &= 2x(3x + 1) + (-5)(3x + 1)\\ &= (2x - 5)(3x + 1)\end{aligned}$$

That is, each of the expressions is equivalent to the remaining expressions because each real number replacement for x results in the expressions having the same value.

There are expressions which are not equivalent for each substitution from the set of real numbers but which are equivalent on a restricted set.

Example 3-2-3

$\frac{x}{x}$ and $\frac{x-3}{x-3}$ are not equivalent for each substitution from the set of real numbers. For $x = 0$, $\frac{x}{x}$ has infinitely many meanings, i.e. is indeterminate and $\frac{x-3}{x-3} = 1$. For $x = 3$, $\frac{x}{x} = 1$ and $\frac{x-3}{x-3}$ is meaningless. However, for every other real number substitution $\frac{x}{x}$ and $\frac{x-3}{x-3}$ become names for the same number. Therefore, we say that $\forall x,\ x \neq 0$ or 3, $\frac{x}{x} = \frac{x-3}{x-3}$, or that $\frac{x}{x}$ and $\frac{x-3}{x-3}$ are equivalent expressions on the set of all real numbers except 0 and 3.

When referring to equivalent expressions we will need to exclude from the replacement set any real numbers which do not produce names for the same numbers.

Example 3-2-4

$\frac{x+2}{x+2}$, $\frac{x-1}{x-1}$ are equivalent on the replacement set $\{x \mid x \in R,\ x \neq -2,\ x \neq 1\}$

Exercise 3-2-5

State the replacement set for which the following expressions are equivalent:

1. $\frac{x}{x}$; $\frac{x - 2}{x - 2}$
2. $3x + 4x - 7$; $7(x - 1)$
3. $\frac{-3}{3x}$; $\frac{1}{x}$
4. $\frac{x - 2}{x - 2}$; $\frac{2 - x}{2 - x}$
5. $\frac{a - b}{c - d}$; $\frac{b - a}{d - c}$
6. $(x - 9)(x + 3)$; $x^2 + (3 - 9)x + (-9)(3)$
7. $(x - 7)(x - 7)$; $x^2 - (7 + 7)x + 49$
8. $acx^2 + (bc + ad)x + bd$; $(ax + b)(cx + d)$
9. $(x - a)(x + a)$; $x^2 - a^2$
10. $(x + c)^2$; $x^2 + 2cx + c^2$
11. $(ax - b)^2$; $ax^2 - 2abx + b^2$
12. $((ax)^3 - b^3)$; $(ax - b)((ax)^2 - abx + b^2)$

3-3 Sentences

Consider the following definitions.

Definition 3-3-1 Equations and Inequations

A sentence consisting of an equality sign (=) with expressions appearing on both sides is an equation. A sentence consisting of an inequality sign ($<, >, \leq, \geq$,) with expressions appearing on both sides is an inequation.

Definition 3-3-2 Sentences

Equations or inequations which do not contain variables are either true or false.
An equation or inequation containing a variable is an open sentence and is neither true nor false.

Example 3-3-3

$$3 + 2 = 5$$

$$7 + -6 = 11$$

$$3a - 5 = 7$$

$$b - (3a + b) = 3(-a)$$

$$-2 < 18$$

$$-50 > -27$$

$$3m \leq -6$$

$$\frac{x}{x} = 1$$

An analysis of those sentences listed in Example 3-3-3 shows that the first two sentences in each column are either true or false while the other sentences are open sentences.

Note that the second sentence in the first column is an equation and the second sentence in the second column is an inequation even though each is false.

Exercise 3-3-4

Identify each of the following sentences as being true or false or open sentences.

1. $c = \pi r^2$
2. $-6 \cdot 1 = 1 \cdot 6$
3. $-8 = -8 \cdot b$
4. $8 + 16 \leq 50$
5. $-13 > |-50|$
6. $a + b \leq |a + b|$
7. $27 = \pi(3)^2$
8. $3(x + 2) \neq 9(y - 8)$

There are two more definitions we must present before turning our attention to the problems encountered when solving equations and inequations.

Definition 3-3-5 Solution Set

The solution set of an equation or inequation is the set of replacements for which the sentence is true.

Example 3-3-6

The solution set of $x^2 + 6x + 8 = 0$ is $\{-4, -2\}$ since $(-4)^2 + 6(-4) + 8 = 0$ and $(-2)^2 + 6(-2) + 8 = 0$ and there are no other replacements which result in a true sentence.

The validity of the above statement may be more readily seen by factoring.

$$\begin{aligned} x^2 + 6x + 8 &= x^2 + (4 + 2)x + 8 \\ &= x^2 + 4x + 2x + 8 \\ &= x(x + 4) + 2(x + 4) \\ &= (x + 2)(x + 4) \end{aligned}$$

Therefore, $(x + 2)(x + 4) = 0$ by the substitution rule. The replacements which make this sentence true are -2 and -4.

Example 3-3-7

The solution set of $\frac{x - 3}{x - 3} = 7$ is $\emptyset$ since $\frac{x - 3}{x - 3}$ and 1 are equivalent on $\{x | x \varepsilon R, x \neq 3\}$ and when $x = 3$, $\frac{x - 3}{x - 3}$ is meaningless.

Example 3-3-8

The solution set of $x + 19 < -5$ is $\{x | x \varepsilon R, x < -24\}$.

Definition 3-3-9 Equivalent Sentences.

Two sentences are equivalent if and only if they have the same solution set.

Example 3-3-10

Consider the equations (1) $2x = -2$ and (2) $x + 1 = 0$. By inspection we see that the solution set of equation (1) is $\{-1\}$. To see if $\{-1\}$ is also of the solution of equation (2) we replace the variable by -1.

$$x + 1 = 0$$

$$(-1) + 1 = 0$$

$$0 = 0$$

So, we see that (-1) is the solution of $x + 1 = 0$. Since, (1) and (2) have the same solution set they are equivalent equations.

Example 3-3-11

Determine whether or not the equations $x^2 + -4 = 0$ and $2x = -4$ are equivalent.

$$\begin{aligned}(1)\ x^2 + -4 = 0 &\leftrightarrow x^2 + (2 + -2)x + -4 = 0\\ &\leftrightarrow x^2 + 2x + -2x + -4 = 0\\ &\leftrightarrow x(x + 2) + -2(x + 2) = 0\\ &\leftrightarrow (x + -2)(x + 2) = 0\end{aligned}$$

Therefore, the solution set of $x^2 - 4$ is $\{2,-2\}$. Since, 2 does not satisfy $2x = -4$, these two equations are not equivalent.

Exercise 3-3-12

Determine whether or not the given pairs of equations are equivalent.

1. $7(a - 2) + 6(1 - a) = 3$; $a = 11$
2. $x^2 + 2x - 3 = 0$; $(x + 3)(x - 1) = 0$
3. $x(x - 2) = 0$; $x = 2$
4. $x^2 - 9 = 0$; $2x = 6$
5. $3x + 4 - 2(2x + 1) \leq 3$; $x \leq -1$

3-4 Solving Equations

We now have all of the definitions and theorems needed to begin the process of solving an equation in one variable.

Consider the Equation:

$$(1)\ 3(5a - 7) - 7(2a - 5) - 14 = 37$$

As you have learned in previous courses the best way to solve this equation is to replace it with another equation which has the same solution set and which can be solved by inspection. In other words, find a new equation which is equivalent to this one and has an obvious solution set.

By applying the principles for real numbers to the expression on the left hand side of this equation we can produce equivalent expressions as shown below.

$$3(5a - 7) - 7(2a - 5) - 14 = 15a - 21 - 14a + 35 - 14 = a$$

By the substitution rule we can derive the new equation $a = 37$.

We also could begin with the equation $a = 37$, and by use of the substitution rule, derive the original equation

$$a(5a - 7) - 7(2a - 5) - 14 = 37$$

In other words, we can deductively prove the following biconditional sentence:

$$3(5a - 7) - 7(2a - 5) - 14 = 37 \leftrightarrow a = 37$$

This biconditional sentence tells us that any replacement which makes the equation $a = 37$ a true statement also makes the equation

$$3(5a - 7) - 7(2a - 5) - 14 = 37$$

a true statement and conversely. Therefore, every root of $a = 37$ is a root of $3(5a - 7) - 7(2a - 5) - 14 = 37$. We can see by inspection that the solution set of $a = 37$ is $\{37\}$. This is also the solution set of the equation

$$3(5a - 7) - 7(2a - 5) - 14 = 37.$$

The manipulation above, which we performed to solve the equation

$$(1)\ 3(5a - 7) - 7\ 7(2a - 5) - 14 = 37,$$

is sometimes called transforming by replacement.

Replacement Transformation Principle for Equations (RTP)

$$\forall a \forall b \forall c \; a = b \text{ and } a = c \rightarrow c = b$$

This can be expressed symbolically by the diagram shown in Figure 3-4-1

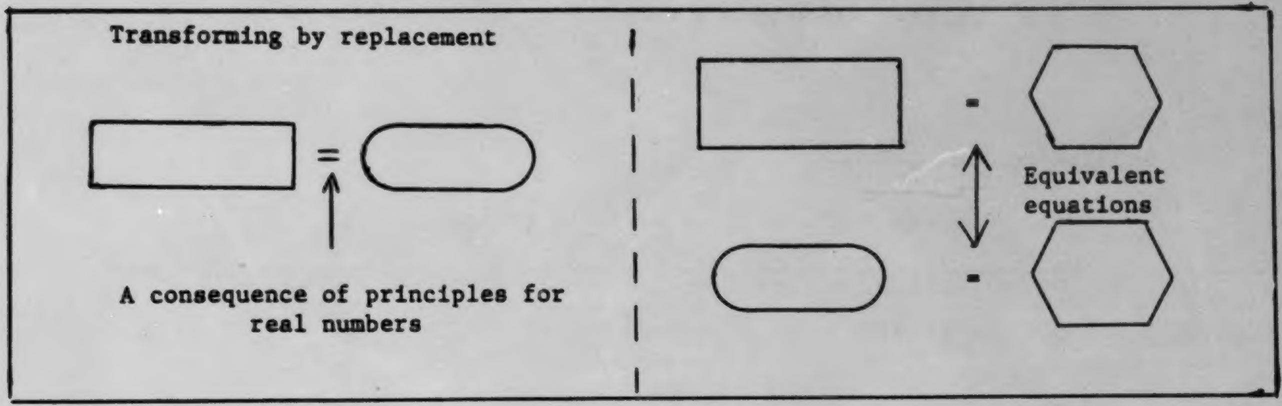
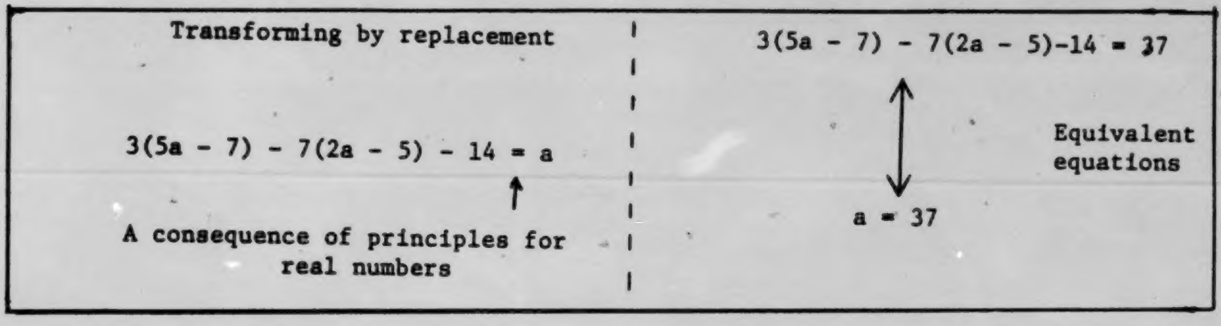

Transforming by replacement
=
A consequence of principles for real numbers
=
Equivalent equations
=

Figure 3-4-1

The following example, Example 3-4-2, shows where the transformation by replacement takes place in Equation (1) above.

Example 3-4-2


Transforming by replacement
3(5a - 7) - 7(2a - 5) - 14 = a
A consequence of principles for real numbers
3(5a - 7) - 7(2a - 5)-14 = 37
Equivalent equations
a = 37

Exercise 3-4-3

Solve each of the following equations and for each one fill in the form shown in Figure 3-4-1 as illustrated in Example 3-4-2.

1. $2(x - 7) + 3(-x + 6) - 4 = 9$

2. $3t - 14 - (2t + 6) + 20 = -3$

The usual procedure for solving the equations in Exercise 3-4-3 is shown below. In each case the justification for producing the final equation is the replacement transformation principle for equations.

Illustration 3-4-3-1

1. $2(x - 7) + 3(-x + 6) - 4 = 9$	1. **Assumption**
2. $2(x - 7) + 3(-x + 6) - 4 = 2x - 14 + -3x + 18 - 4$	2. **Equivalent Expressions**
3. $2x - 14 + -3x + 18 - 4 = -x$	3. **Equivalent Expressions**
4. $-x = 9$	4. **Replacement**

Illustration 3-4-3-2

1. $3t - 14 - (2t + 6) + 20 = -3$	1. Assumption
2. $3t - 14 - (2t + 6) + 20 = t$	2. Equivalent Expressions
3. $t = -3$	3. Replacement

Not all equations can be solved by application of the replacement transformation principle for equations. Consider the equation:

$$(1)\ 5a + 9 = 13 - 2a$$

It is difficult to solve this equation by inspection. One could try guessing.

Try 7.

$$5 \cdot 7 + 9 \overset{?}{=} 13 - 2 \cdot 7$$

$$44 \neq -1$$

Try 2.

$$5 \cdot 2 + 9 \overset{?}{=} 13 - 2 \cdot 2$$

$$19 \neq 9$$

Try 1.

$$5 \cdot 1 + 9 \overset{?}{=} 13 - 2.1$$

$$14 \neq 11$$

Try 0.

$$5 \cdot 0 + 9 \overset{?}{=} 13 - 2 \cdot 0$$

$$9 \neq 13$$

It seems that a root is between 0 and 1. Why? Is it closer to 0 than to 1? You might get to it by continuing this process of guessing and "closing in" on it. But there is a much faster procedure.

What we need to do is find an equation which is equivalent to (1) and which is easier to solve. What is there about (1) which makes it difficult to solve? Do you see how (1) can be transformed into an equivalent equation in which the variable occurs in only on side?

Here is one possibility. Add 2a to both sides of (1) (that is, put a '+2a" on each side) and write the new equation.

$$(1)\ 5a + 9 = 13 - 2a$$

$$(2)\ 5a + 9 + 2a = 13 - 2a + 2a$$

Equation (2) has the variable occurring in both sides but we can transform it by replacement into:

$$(3)\ 7a + 9 = 13$$

and (3) is fairly easy to solve by inspection. If (1) and (3) are equivalent, once we have solved (3), we shall have solved (1). Are (1) and (3) equivalent? Let's show that they are.

Consider (1) and (2). By an addition principle for equality,

$$\forall a,\ 5a + 9 = 13 - 2a \rightarrow 5a + 9 + 2a = 13 - 2a + 2a$$

By the cancellation principle for addition,

$$\forall a,\ 5a + 9 + 2a = 13 - 2a + 2a \rightarrow 5a + 9 = 13 - 2a$$

So, by these two principles,

$$(4)\quad \forall a,\ 5a + 9 = 13 - 2a \leftrightarrow 5a + 9 + 2a = 13 - 2a + 2a$$

This biconditional sentence tells us that (1) and (2) have the same solution set that is, that (1) and (2) are equivalent equations. The manipulation we performed on equation (1) to get equation (2) is sometimes called transforming by addition.
It may be illustrated symbolically by the diagram shown in Figure 3-4-4.

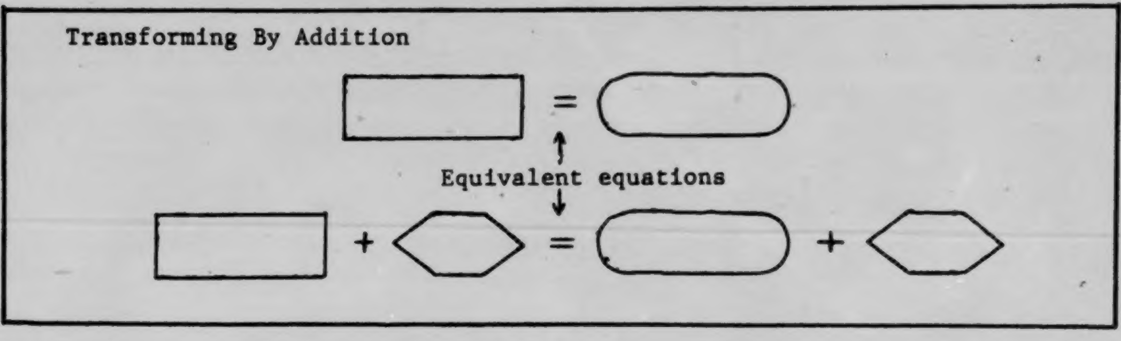


Figure 3-4-4

This manipulation is justified by the following principle:

Addition Transformation Principle for Equations (ATP)

$$\forall x \forall y \forall z \quad x = y \leftrightarrow x + z = y + z$$

This generalized biconditional follows from the addition principle for equality and the cancellation principle for addition.

To complete the job of showing that (1) and (3) are equivalent we use the replacement transformation principle:

1. $5a + 9 = 13 - 2a$	1. Assumption
2. $5a + 9 + 2a = 13 - 2a + 2a$	2. ATP
3. $7a + 9 = 13$	3. RTP(used twice)

Each of the above equations is equivalent to the others. The first use of replacement depends upon the fact that, $5a + 9 + 2a = 7a + 9$ and the second use depends upon the fact that, $13 - 2a + 2a = 13$.

Although the equation '$7a + 9 = 13$' can be solved by inspection, we could solve it by transforming it into an equivalent equation, and continuing the transforming procedure until we reach an equation of the form:

$$a = \;.\;.\;.\;.\;.\;.\;. \leftarrow \text{numeral}$$

Let's do so.

Example 3-4-5

1. $7a + 9 = 13$	1. Assumption
2. $7a + 9 + -9 = 13 + -9$	2. ATP
3. $7a = 4$	3. RTP
4. $7a \cdot \frac{1}{7} = 4 \cdot \frac{1}{7}$	4. (Multiplication; $\frac{1}{7} \neq 0$)
5. $a = \frac{4}{7}$.	5. RTP

Notice the step from 3, to 4. This manipulation is an example of transforming by multiplication. The conditional:

$$7a = 4 \rightarrow 7a \cdot \frac{1}{7} = 4 \cdot \frac{1}{7}$$

is a consequence of a multiplication principle for equality:

$$\forall x \forall y \forall z, x + y \rightarrow xz = yz$$

The conditional's converse:

$$7a \cdot \frac{1}{7} = 4 \cdot \frac{1}{7} \rightarrow 7a = 4$$

is a consequence of '$\frac{1}{7} \neq 0$' and the cancellation principle for multiplication:

$$\forall x \forall y \forall z \neq 0, x = y \leftarrow xz = yz$$

Transforming by multiplication is justified by the following principle.

Multiplication Transformation Principle for Equations (MTP)

$$\forall x \forall y \forall z \neq 0,\; x = y \leftrightarrow xz = yz$$

This generalized biconditional follows from the multiplication principle for equality and the cancellation principle for multiplication. It is illustrated symbolically in Figure 3-4-6.

Now we see that the reason given for Step 4 in Example 3-4-5 should be MTP.

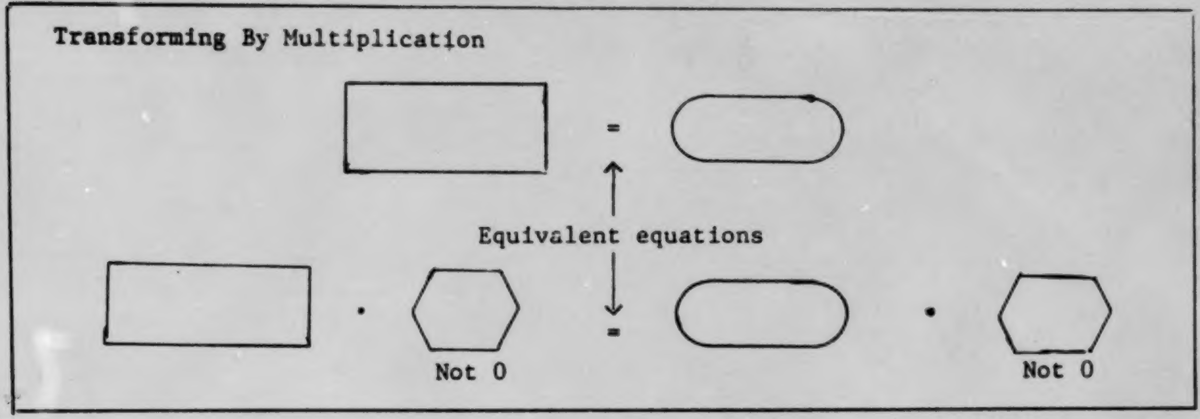


Figure 3-4-6

In solving the equations in the following exercises you should keep in mind that the solving procedure consists of transforming the given equation into a second equation equivalent to the first, transforming the second one into another equivalent equation, and continuing in this manner until you reach an equation whose roots are obvious (variable = numeral). Since each step produces an equation equivalent to the preceding equation, when you reach the final equation you will know that its roots are precisely those of the given equation. However, in order to catch any manipulation errors which you may have committed along the way, it is a good idea to check the roots by substitution in the given equation. As with any manipulation practice, you will find yourself able to leave out more and more intermediate steps as you become more proficient.

Example 3-4-7

Solve: $5a - 3 + 4a + 5 = 20$

$$5a - 3 + 4a + 5 = 20$$

$$9a + 2 = 20$$

$$9a + 2 + -2 = 20 + -2$$

$$9a = 18$$

$$a = 2$$

Check. $(5 \cdot 2) - 3 + (4 \cdot 2) + 5 = 10 - 3 + 8 + 5 = 20$✓

Example 3-4-8

Solve: $8 - 4b = 7 - 9b$

$$8 - 4b = 7 - 9b$$

$$8 - 4b + (9b - 8) = 7 - 9b + (9b - 8)$$

$$5b = -1$$

$$\frac{5b}{5} = \frac{-1}{5}$$

$$b = -\frac{1}{5}$$

Check.

$$8 - (4 \cdot -\frac{1}{5}) = 8 + \frac{4}{5} = \frac{44}{5}$$

$$7 - (9 \cdot -\frac{1}{5}) = 7 + \frac{9}{5} = \frac{44}{5}\checkmark$$

Example 3-4-9

Solve: $\frac{b}{3} + 7 + \frac{b}{5} = 9 + \frac{2b}{3}$

$$\frac{b}{3} + 7 + \frac{b}{5} = 9 + \frac{2b}{3}$$

$$15(\frac{b}{3} + 7 + \frac{b}{5}) = 15\ (9 + \frac{2b}{3})$$

$$5b + 105 + 3b = 135 + 10b$$

$$8b + 105 = 135 + 10b$$

$$8b - 10b = 135 - 105$$

$$-2b = 30$$

$$b = \frac{30}{-2} = -15$$

Check. $\frac{-15}{3} + 7 + \frac{-15}{5} = -5 + 7 - 3 = -1;$

$$9 + \frac{2 \cdot -15}{3} = 9 - \frac{30}{3} = -1\checkmark$$

Note that the multiplication which took place in Example 3-4-9, Step (2) involved the least common multiple of the denominators of the fraction.

Exercise 3-4-10

Solve each of the following equations and check your answers.

1. $7 + -5 = 15 + 3t$
2. $8 - k = 5k - 10$
3. $5b - 3 = 9 + 2b$
4. $3k - 10.5 = 4.5 - 12k$
5. $12 - 11y - 2 + 9y = 0$
6. $3a + 7 + 2a = 6a + 17 + a$

7. $7x + 9 = 4x + 18$

8. $4b - 6(3 - b) = 3b + 1 - 5(b - 1)$

9. $2(x - 3) + 5(x - 4) = 14x - 27$

10. $9 + 2(3 - a) = 12a - 5(a + 3) - 33$

11. $\frac{a}{4} - 5 = \frac{a}{3}$

12. $\frac{t}{4} + \frac{t}{8} = 5 + \frac{t}{3}$

13. $\frac{3b}{4} + 22 = \frac{8b}{7}$

14. $\frac{k}{2} + \frac{3k}{5} - 1 = \frac{k}{10} - \frac{2k}{3}$

15. $\frac{3x}{5} + \frac{x}{7} = \frac{x}{5} + 38$

16. $\frac{5y}{6} - \frac{2y}{13} = 7 + \frac{y}{2}$

17. $\frac{t}{7} - \frac{t}{8} = \frac{t}{8} - \frac{t}{7}$

18. $\frac{3}{5} - a = \frac{a}{4} + \frac{1}{10}$

19. $x - \frac{1}{3} = \frac{2}{5} - x$

20. $5 + \frac{y}{3} = 3 - \frac{y}{5}$

21. $\frac{a}{2} - \frac{a}{4} + \frac{2 - 3a}{6} = 0$

22. $\frac{m + 2}{3} = m + 0.5$

23. $\frac{m + 3}{2} = \frac{m + 7}{5}$

24. $\frac{4a + 1}{3} = \frac{5 - a}{6}$

25. $\frac{x}{4} + \frac{x + 2}{2} = 10$

3-5 Additional use of the Multiplication Transformation Principle for Equations.

In the preceding section we developed the mulitplication transformation principle for equations:

$$\forall x \forall y \forall z \neq 0, x = y \leftrightarrow xz = yz$$

Following are some examples of use of this transformation principle.

Example 3-5-1

Solve: $9 - \frac{3a + 2}{a} = \frac{-7}{a}$

$$\left(9 - \frac{3a + 2}{a}\right)a = \left(-\frac{7}{a}\right)a, \quad (a \neq 0)$$

$$9a - 3a - 2 = -7$$

$$6a = -5$$

$$a = -\frac{5}{6}$$

Check.
$$9 - \frac{3(-\frac{5}{6}) + 2}{-\frac{5}{6}} = \frac{-7}{-\frac{5}{6}}$$

$$9 - \frac{-\frac{5}{2} + \frac{4}{2}}{-\frac{5}{6}} = \frac{42}{5}$$

$$9 - \frac{-\frac{1}{2}}{-\frac{5}{6}} = \frac{42}{5}$$

$$9 - \frac{6}{10} = \frac{42}{5}$$

$$\frac{45}{5} - \frac{3}{5} = \frac{42}{5} \checkmark$$

Since, $-\frac{5}{6} \neq 0$ and $-\frac{5}{6}$ satisfies the given equation it is the solution of $9 - \frac{3a + 2}{a} = \frac{-7}{a}$.

Example 3-5-2

$$4 - \frac{3b - 5}{b} = \frac{5}{b}$$

$$(4 - \frac{3b - 5}{b})b = (\frac{5}{b})\ b, \quad (b \neq 0)$$

$$4b - (3b - 5) = 5$$

$$b + 5 = 5$$

$$(b + 5) + -5 = 5 + -5$$

$$b = 0$$

The solution set of the equation $b = 0$ is $\{0\}$. Is this the solution set of the original equation? Note the first step in transforming the original equation into one which is easier to solve by inspection. In this step the MTP was used with the restriction $b \neq 0$. This tells us that the derived equation and the original equation have the same roots in the set of all non-zero real numbers. What about zero? We don't know if it is a root of the original equation. We must find out by substitution. Substitution shows that zero is not a root of the original equation. This example shows that sare must be exercised in the use of the MTP in solving equations.

Example 3-5-3

$$x^2 - x - 6 = 6(x - 3)$$

$$(x + 2)(x - 3) = 6(x - 3)$$

$$(x + 2)(x - 3)\frac{1}{x - 3} = 6(x - 3)\frac{1}{x - 3}, \quad -x - 3 \neq 0$$

$$x + 2 = 6 \quad (x - 3 \neq 0)$$

$$x = 4$$

The roots of the derived equation, $x = 4$, are the same as those of the original equation in the set of all real numbers except 3. (Explain why.) This means that 4 is a root of $x = 4$ and $x^2 - x - 6 = 6(x - 3)$ in the set of all real numbers that are not 3. We do not know whether 3 is a root of the original equation or not without testing to see. Substitution shows that 3 is a root of the original equation.

Therefore, the solution set of the equation $x^2 - x - 6 = 6(x - 3)$ is $\{3,4\}$

This again illustrates that care must be exercised in deriving equations which are easier to solve than the original equation. Often derived equations are equivalent to the original equations on restricted replacement sets. Numbers not included in the restricted set must be checked by substitution in the original equation.

Exercise 3-5-4

Solve the following equations and check your answers.

1. $2x(x - 7) = x^2 + 4x$
2. $3x(x - 4) = x(x + 2)$
3. $3 - \frac{2y - 8}{y} = \frac{8}{y}$
4. $8 + \frac{3x - 7}{x} = \frac{15}{x}$

5. $a + 4 + \frac{16}{a + 2} = \frac{a^2}{a + 2}$

6. $x^2 - x - 6 = x - 3$

7. $a^2 + a - 56 = a + 8$

8. $y^2 + 8y + 16 = y(y + 4)$

9. $\frac{a^2 - 12a + 36}{a - 6} = 0$

10. $x^2 - 2x - 15 = \frac{2x^2 - 4x - 30}{2}$

11. $8x - 56 = 3x^2 - 19x - 14$

12. $\frac{x + 2}{2} + \frac{3x}{5} + \frac{x + 1}{4} = 16$

13. $\frac{b + 11}{6} + \frac{b - 10}{3} = 1$

14. $x + \frac{x + 1}{2} + \frac{4x}{5} = 58$

15. $\frac{3}{2x + 1} = \frac{8}{4x + 3}$

16. $- \frac{16}{x - 4} = \frac{4x}{4 - x} - x$

17. $\frac{180}{\frac{3}{2}x} = \frac{180}{x} - 2$

18. $\frac{2x^2}{2x - 1} - 3x + (-1) = \frac{x}{2x - 1}$

19. $\frac{7x}{x - 6} = \frac{72 - 5x}{x - 6} - 2x$

20. $\frac{5}{2} \cdot \frac{(1)}{x - 1} = \frac{39x - 3x}{2(x - 1)} - \frac{2}{3}$

3-6 Coding and Solving Equations by Computer.

One form of a general linear equation in one variable is $Ax + B = Cx + D$. By using the equation transformation principles, we can solve for x in terms of A, B, C, and D.

$$Ax + B = Cx + D$$

$$Ax + B + -Cx = Cx + D + -Cx$$

$$Ax + -Cx + B = Cx + -Cx + D$$

$$Ax + -Cx + B = 0 + D$$

$$(A + -C)x + B + -B = D + -B$$

$$(A + -C)x = (D + -B)$$

$$x = \frac{D + -B}{A + -C}$$

Careful observation reveals that the solution of the equation is a rational expression (fraction) whose numerator is the sum of the constant on the right of the equals symbol and the opposite (additive inverse) of the constant on the left. The denominator is the sum of the coefficient of the variable on the left and the opposite of the coefficient of the variable on the right.

These ideas can be applied to a specific equation: $3x + 2 = 4x - 7$. The process for solving this equation is as follows:

$$3x + 2 = 4x - 7$$

$$3x + 2 + -2 = 4x - 7 + -2$$

$$3x + 0 = 4x - 7 + -2$$

$$3x + -4x = 4x - 7 + -2 + -4x$$

$$(3 + -4)x = 0 - 7 + -2$$

$$x = \frac{-7 + -2}{3 + -4}$$

The simplified form of the fraction $\frac{-7 + (-2)}{3 + -4}$ is unimportant at the moment. What is important is the relation between the position of each numeral in the equation $(3x + 2 = 4x - 7)$ and the position of the numerals in the expression, $\frac{-7 + -2}{3 + -4}$. The numerator $(-7 + -2)$ contains the constant on the right (-7) and the opposite of the constant $(+2)$ on the left of the equation. The denominator contains the coefficient (3) of the variable on the left and the opposite of the coefficient (4) of the variable on the right of the equation. This idea can be extended to determine the solution of any linear equation in one variable.

Exercise 3-6-1

In these exercises write a fraction which is the solution. Do so by inspection.

1. $2x - 5 = x + 1$
2. $3x + 1 = 7 + x$
3. $5x - 1 = 6x + 2$
4. $4x = x - 6$
5. $5x - 8 = x$
6. $6x + -2 + 4x - 3x = 29 - 7x + -3$

The logic flow chart in Figure 3-6-2 shows the sequential steps in deciding whether to add a numerical coefficient or a constant to the numerator or the denominator. It also determines when the inverse of these numbers should be used.

If you had difficulties with the previous exercises, use this flow chart to help you.

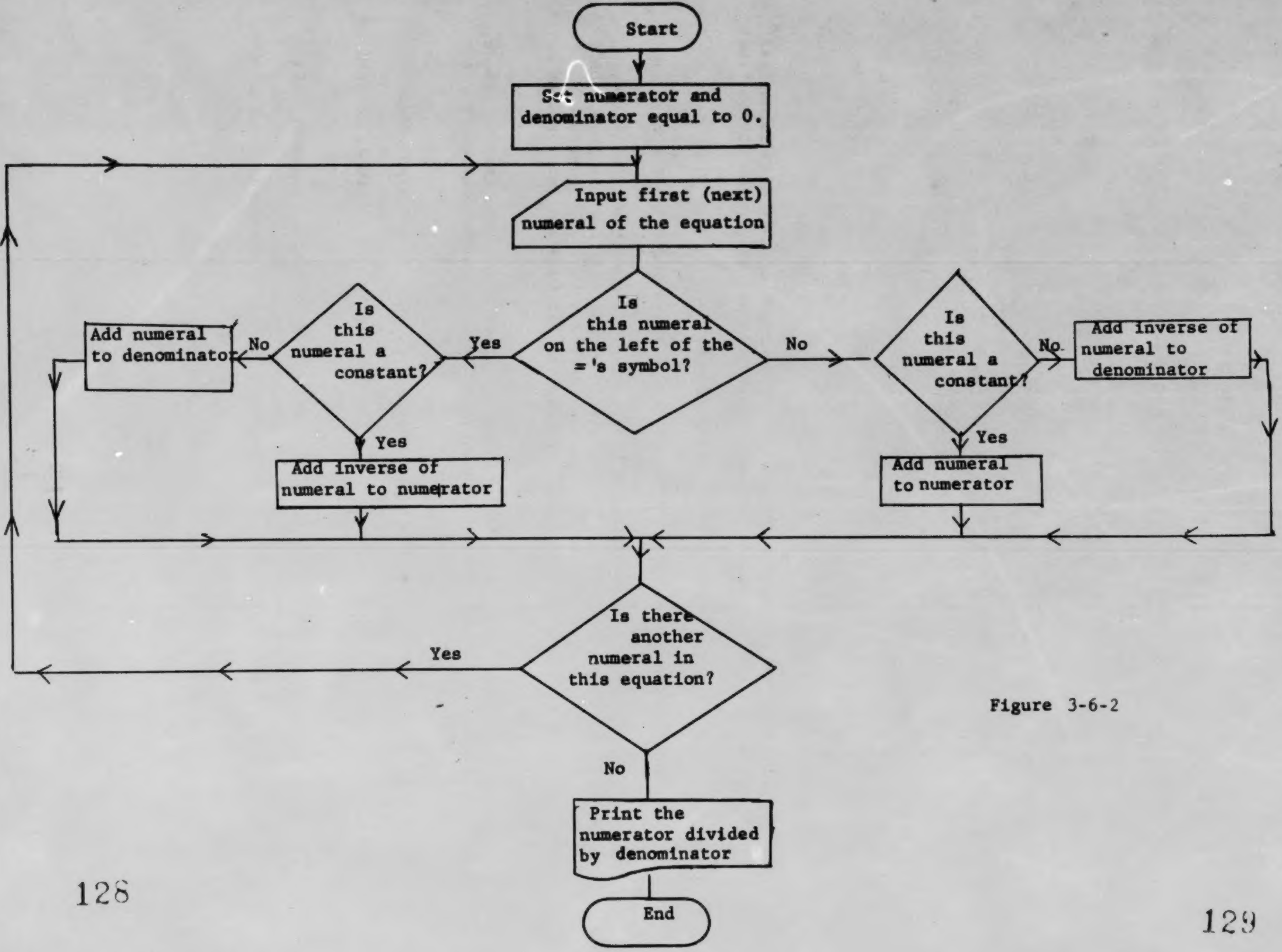

Start
Set numerator and denominator equal to 0.
Input first (next) numeral of the equation
Is this numeral on the left of the ='s symbol?
Yes
No
Is this numeral a constant?
No
Add numeral to denominator
Yes
Add inverse of numeral to numerator
Is this numeral a constant?
No
Add inverse of numeral to denominator
Yes
Add numeral to numerator
Is there another numeral in this equation?
Yes
No
Print the numerator divided by denominator
End

Figure 3-6-2

Exercise: 3-6-3

Write an equation for each of the following solutions in the form ax + b = cx + d.

1. $\{\frac{7 + (-3)}{8 + 6}\}$
2. $\{\frac{6}{2 + (-5)}\}$
3. $\{\frac{-4 + 1}{9 + 5 - 7}\}$
4. $\{\frac{-5 - 2}{-7 - 1}\}$
5. $\{\frac{0}{3 - 2}\}$

Writing a program for the computer to solve any equation of this kind is our next objective. Keep in mind that the computer can handle an instruction of the type:

```
70 LET X = (D - B) / (A - C)
```

This command or some variation of it may be useful as we begin to program.

There is a difficulty to be overcome in programming the computer to solve any linear equation in one variable. The computer can read only numbers from a DATA statement. It cannot read an equation such as 3x + 2 = 4x - 7. Therefore, we must develop a numerical code which the computer can read and interpret as an equation.

In the equation 3x + 2 = 4x - 7 the term "3x" occupies a position that is unique. No other term is in exactly the same location. There is a numerical coefficient, 3, in this term.

The same is true of the term "4x". The constants, 2 and -7, are also unique in location and type of term. Keep these facts in mind. We will now develop a code which will enable us to interpret for the computer where a term is located and if it is a variable term or constant term.

If a term appears on the left of the equals symbol, a "0" might represent this fact. It would follow then that a term on the right side could be coded with a "1". We could similarily code a constant with "0" and a numerical coefficient of a variable with a "1".

Using these ideas we could code the equation as follows:

Equation:	3x	+2	=	4x	-7
Code:	0,3,1	0,2,0		1,4,1	1,-7,0

Each group of three numbers in the code represents a term of the equation.

For example, in the first three numbers of the code (0, 3, 1), the "0" tells us we are referring to a term on the left, the "3" indicates the term contains a numeral 3 and the "1" tells us that the 3 is a coefficient of a variable.

We could use the above code as data to communicate the equation to the computer.

Exercise: 3-6-4

1. Code the following equations:

   a. $5x - 6 = 2x + 8$

   b. $x + 3 = 7$

   c. $6x - 2x + 3 = 0$

   d. $-11x + 5 = 3 + 4x$

2. Write equations from the following coding:

   a. 0, 3, 0, 1, 1, 1, 0, -5, 1, 1, 2, 0

   b. 1, -4, 0, 0, 0, 0, 0, 3, 0, 1, 1, 1

   c. 0, -8, 1, 0, 2, 0, 1, 3, 1, 1, 1, 0

   d. 0, 4, 1, 1, -2, 1, 0, -3, 0, 1, 0, 0

At this point we are ready to look at a possible program for solving any linear equation in one variable. Read the partial program through carefully. Note the new commands.

```
  10 DATA 0, 3, 1, 0, 7, 0, 1, 3, 1, 1, 2, 0
  20 LET N = 0
  30 LET D = 0
  40 READ A, B, C
  50 ØN A + 1 GØ TØ 100, 200
 100 ØN C + 1 GØ TØ 300, 400
 200 ØN C + 1 GØ TØ 500, 600
 300 LET N = N + (-B)
 350 GØ TØ 40
 400 LET D = D + B
 450 GØ TØ 40
 500 LET N = N + B
 550 GØ TØ 40
 600 LET D = D +(-B)
 650 GØ TØ 40
 700
   .
   .
   .
1000 END
```

In any application of the ON---GO TO instruction, the following format must be used.

ØN (expression) GØ TØ line number, line number, line number, line number, ---

When the value of the expression is 1, the computer will jump to the program line that is numbered in the first position following the words GØ TØ. When the value of the expression is 2, the computer will jump to the program line that is numbered in the second position following the GØ TØ. In general if the value of the expression is n, the computer will jump to the program line that is numbered in the nth position after the GØ TØ. The only constraint in the use of this instruction is that there can be no more than 15 line numbers following the GØ TØ.

Figure 3-6-5

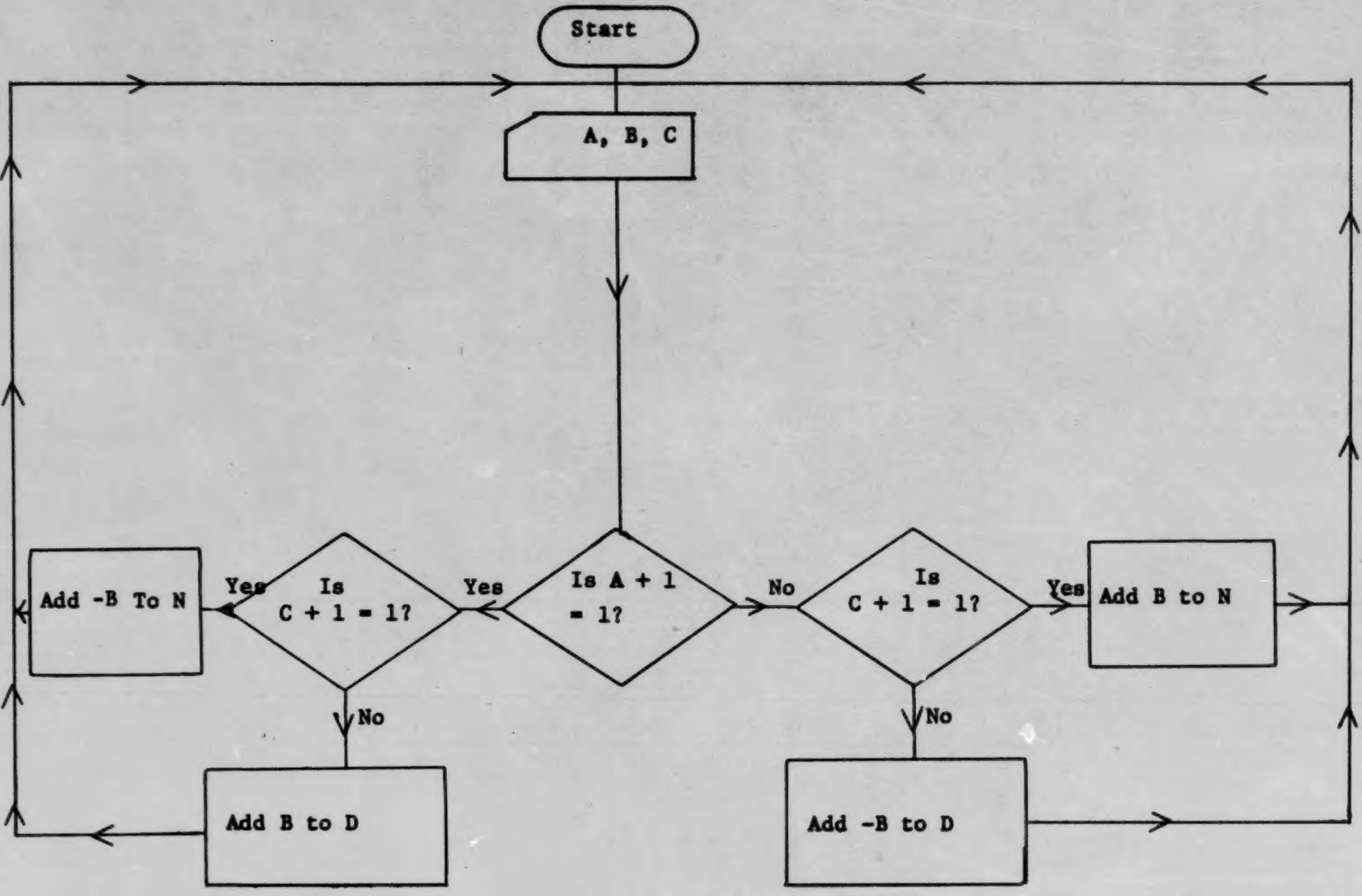


Figure 3-6-5 is a flow chart for the "ØN - GØ TØ" part of this program. It is not a complete flow chart.

The new command ØN - GØ TØ enables us to direct the computer to more than one succeeding line in the program. Look again at the program. Lines 10 and 40 will make the first value of A zero. Line 50 tells the computer to add 1 to this value of A. In this case 0 + 1 = 1. Since A + 1 = 1 the computer goes to the first number following the words GØ TØ. This number is 100. It is the line number of the next instruction that the computer will execute. If the computer had read an A having a value 1, then A + 1 = 2. Now the computer would go to the second number (200) following the GØ TØ part of the command. This number would be the line number of the next command to be executed.

Assume that the computer has been directed to line 100. In line 100 the computer uses values for C read in the data statement. The first value for C in this data statement, is 1. Since C = 1, then C + 1 = 2 and the computer will be directed to line 400, the line number in the second position after GØ TØ. If C had been equal to 0, then C + 1 = 1, and the computer would be directed to line 300, the line number in the first position after GØ TØ.

The computer could have been directed to line 200 if A + 1 = 2 in line 50. If so, in line 200 after reading a C in the data, C + 1 would be evaluated and the computer would jump to either line 500, or 600 depending on the value of C + 1.

Lines 50, 100, and 200 of the program contain the logic branching "ØN--GØ TØ" instructions which direct the computer to one of the lines 300, 400, 500, or 600. This logic branching depends on the values of A and C as described above. In line 300 the computer adds the opposite of a constant on the left to the numerator N. In line 400 the computer adds a coefficient on the left to the denominator D. In line 500 the computer adds a constant on the right to the numerator N. In line 600 the computer adds the opposite of a coefficient on the right to the denominator. Hence after all values of A, B, and C have been processed by the computer, N will be the numerator and D will be the denominator for the solution of the equation.

The program as written allows the computer to read data from line 10 through four loops of the program. If we were to run the program as it stands, it would print "ØUT ØF DATA IN # 40" and terminate. In order to direct the program to continue to solve the equation we must insert two additional commands. First, another data statement, 11 DATA 9999999, 0, 0 makes the computer read a "dummy" set of information.

The second command, 45 IF A = 9999999 THEN 700 tells the computer that the job of reading pertinent data is done; now move to the next phase of the program which begins in line 700. This is known as a "flag" technique; the flag, in this case is 9999999. This number is chosen because it is extremely unlikely that it would be used in coding an equation. (In this coding system only 0 or 1 will appear in the first location, A).

```
 10 DATA 0, 3, 1, 0, 7, 0, 1, 3, 1, 1, 2, 0

 11 DATA 9999999, 0, 0

 20 LET N = 0

 30 LET D = 0

 40 READ A, B, C

 45 IF A = 9999999 THEN 700

 50 ØN A + 1 GØ TØ 100, 200

100 ØN C + 1 GØ TØ 300, 400

200 ØN C + 1 GØ TØ 500, 600

300 LET N = N  + (-B)

350 GØ TØ 40

400 LET D = D + B

450 GØ TØ 40

500 LET N = N + B

550 GØ TØ 40

600 LET D = D + (-B)

650 GØ TØ 40

700  .   .   .

 .

 .

 .

1000 END
```

Figure 3-6-6

Exercise 3-6-7

1. Finish writing the program shown in Figure 3-6-6 so the computer will print the solution set for any linear equation in one variable which is properly coded in line 10.

2. Use the following as data for the program you wrote in #1 above and solve the equations represented.

   a. 0, 7, 1, 0, 3, 0, 1, 2, 1, 1, 1, 0

   b. 0, 2, 1, 0, 1, 0, 1, 2, 1, 1, 3, 0

   c. 0, 10, 1, 1, 8, 1, 0, 3, 0, 1, 7, 0, 1, -4, 0

   d. 0, -3, 0, 0, 2, 1, 0, -4, 0, 0, 11, 1, 1, 7, 1, 1, -7, 0, 1, 6, 1

3. If your program does not produce the correct solution set for each equation, go back and do part 1 over.

Problem Set 3-6-8

1. Expand your program to reduce the fractional solution to lowest terms.

* 2. Incorporate into the program the instructions necessary in order for the computer to print out the equation along with its solution set.

3-7 Solving Inequations

In applications of mathematics we deal with many quantities that are unequal. Therefore, it is important that we study the inequality relationship between pairs of real numbers and learn how to solve inequations.

The inequality relations, less than and greater than, are defined as follows:

Definition 3-7-1 Less than.

$$\forall a \forall b,\ a < b \leftrightarrow \exists \text{ a positive number } p$$

$$\text{such that } a + p = b$$

Definition 3-7-2 Greater than.

$$\forall a \forall b, a > b \leftrightarrow b < a$$

The inequality relation "less than" satisfies one of three equivalence relations. This relation will be considered our first axiom of inequalities.

$$\forall a \forall b \forall c \text{ If } a < b \text{ and } b < c \text{ then } a < c \text{ (Transitivity)}$$

It is interesting to note that the inequality relation, less than, is neither reflexive nor symmetric. That is, it is not true that $\forall a\ a < a$ (Reflexive). Neither is it true that $\forall a \forall b\ a < b \leftrightarrow b < a$ (Symmetric).

Another important relationship between pairs of real numbers is that of trichotomy which is stated below.

$$\forall a \forall b \text{ Exactly one of the following is true:}$$

$$a < b,\ a > b, \text{ or } a = b \text{ (Trichotomy)}$$

The two definitions above and the properties we developed for equations are sufficient for solving most inequations.

Example 3-7-3: Solve $x + 3 < 7$

1. $x + 3 < 7$	1. Assumption
2. $x + 3 + p = 7,\ p > 0$	2. Definition of less than
3. $x + p = 4$	3. APE and simplification
4. $x < 4$	4. Definition of less than

Example 3-7-4: Solve $2x + 3 < 6$

1. $2x + 3 < 6$	1. Assumption
2. $2x + 3 + p = 6,\ p > 0$	2. Definition of less than
3. $2x + p = 3$	3. APE and simplification
4. $x + \frac{p}{2} = \frac{3}{2}$	4. MPE and simplification
5. $\frac{p}{2} > 0$	5. $\forall x > 0 \forall y > C, \frac{x}{y} > 0$
6. $x < \frac{3}{2}$	6. Definition of less than

Example 3-7-5: Solve $4x + 1 > 7 + 2x$

1.	$4x + 1 > 7 + 2x$	1.	**Assumption**
2.	$7 + 2x < 4x + 1$	2.	**Definition of greater than**
3.	$7 + 2x + p = 4x + 1, (p > 0)$	3.	**Definition of less than**
4.	$6 + p = 2x$	4.	**APE and simplification**
5.	$3 + \frac{p}{2} = x$	5.	**MPE and simplification**
6.	$\frac{p}{2} > 0$	6.	$\forall x > 0, \forall y > 0, \frac{x}{y} > 0$
7.	$3 < x$	7.	**Definition of less than**
8.	$x > 3$	8.	**Definition of greater than**

Exercise 3-7-6

Using the inequality definitions and properties of equality solve the following:

1. $x + 3 > 6$
2. $2x - 1 < 3$
3. $-2x > x$
4. $x - 7 \nless 5 - 3x$
5. $-7x + 4 > 3x - 2$
6. $-2x + 3 < 0$

Now we will prove some theorems which will facilitate solving inequations.

Theorem 3-7-7: Additional Transformation Principle of Inequality. (ATPI)

$$\forall a \forall b \forall c \; a < b \leftrightarrow a + c < b + c$$

Proof:

1.	$a < b$	1.	**Assumption**
2.	$a < b \rightarrow \exists$ a positive number p such that $a + p = b$	2.	**Definition 3-7-1**
3.	$\exists$ a positive number p such that $a + p = b$	3.	**(1) (2) Modus Ponens**
4.	$a + p = b \rightarrow a + p + c = b + c$	4.	**APE**
5.	$a + p + c = b + c$	5.	**(3) (4) Modus Ponens**
6.	$a + p + c = a + (p + c) = (a + c) + p$	6.	**APA, CPA**
7.	$(a + c) + p = b + c$	7.	**(5) (6) Substitution Rule**
8.	$(a + c) + p = b + c$ and p is a positive number $\rightarrow a + c < b + c$	8.	**Definition 3-7-1**
9.	$a + c < b + c$	9.	**(7) (8) Modus Ponens**
10.	$a < b \rightarrow a + c < b + c$	10.	**Deduction Rule**

The theorem $\forall a \forall b \forall c \; a + c < b + c \rightarrow a < b$ could be proven in a similar manner.

These two theorems may be combined as follows:

$$\forall a \forall b \forall c \; a < b \leftrightarrow a + c < b + c$$

Exercise 3-7-8: Prove the following theorem, 3-7-9.

Theorem 3-7-9: Multiplication Transformation Principle for Inequalities (MTPI)

$$\forall a \forall b \forall c > 0, \; a < b \leftrightarrow ac < bc$$

$$\forall a \forall b \forall c < 0, \; a < b \leftrightarrow ac > bc$$

The theorem you have just proved emphasizes the only principle for solving inequations which differs from the Transformation Principles for equations. That is when the Multiplication Transformation Principle is applied to an inequation care must be taken if the factor used is a negative number. If this factor is less than zero the order of the inequality is changed. See Example 3-7-11, below.

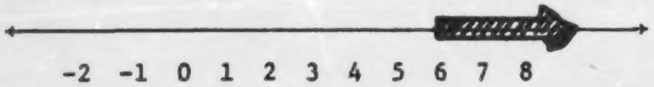
These transformation principles for inequalities can also be used to transform sentences containing $\geq$ or $\leq$ symbols as shown in the following examples.

Example 3-7-10:

Find the solution set of $3x + 17 \geq 35$.

1. $3x + 17 \geq 35$	1. Assumption
2. $3x > 18$	2. ATPI and simplification
3. $x \geq 6$	3. MTPI and simplification

The solution set for this inequation is $\{x \mid x \geq 6\}$.
The graph of the solution set follows.



Example 3-7-11:

Find the solution set of $-4y + 6 \leq 7 - y$.

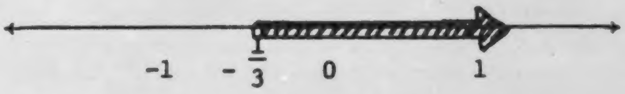
1. $-4y + 6 \leq 7 - y$ 1. Assumption

2. $-3y \leq 1$ 2. ATPI and simplification

3. $y \geq -\frac{1}{3}$ 3. MTPI and simplification

The solution set for this inequation is $\{y | y \geq -\frac{1}{3}\}$

The graph of the solution set follows.



Sometimes inequations are written in the form $a < b < c$.

Definition 3-7-12:

$\forall a,b,c \in R,\ a < b < c \leftrightarrow a < b$ and $b < c$.

Let us now solve an inequation of this type.

Example 3-7-13: Solve $7x < x - 2 < 3$

1. $7x < x - 2 < 3$ 1. Assumption

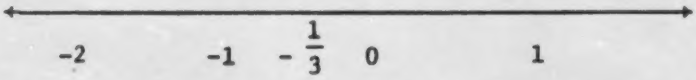
2. $7x < x - 2$ and $x - 2 < 3$ 2. Definition of $a < b < c$

3. $6x < -2$ and $x < 5$ 3. ATPI and simplification

4. $x < -\frac{1}{3}$ and $x < 5$ 4. MTPI and simplification

The solution set is $\{x | x < -\frac{1}{3}\} \cap \{x | x < 5\} = \{x | x < -\frac{1}{3}\}$

The graph of the solution set follows.



Exercise 3-7-14:

Find the solution set for each of the following inequations. Graph each solution set.

1. $2x - 1 < 7$
2. $3x + 2 \geq 5$
3. $-2x > x$
4. $-4x + 7 \leq 3 - 2x$
5. $2x + 5 < 6x - 13$
6. $3 \leq \frac{3}{5}x - 6$
7. $\frac{x}{3} + \frac{2}{5} > \frac{3x}{2} + 1$
8. $\frac{2b - 5}{2} > \frac{3b + 4}{3}$
9. $\frac{3c - 4}{4} \leq \frac{c + 1}{3} - 2$
10. $2 < 4x < x + 9$
11. $x + 1 < 2x - 1 < 3 + x$
12. $3 - 2x \geq 7x \geq x + 5$
13. $3x + 8 \leq x \leq 3 + 2x$
14. $4x - 1 < 3x + 2 < -3 + 4x$

There is one other transformation principle which you should learn now. To see why, let's try to solve:

$$a^2 + 4 > 5(2 - a) \qquad (1)$$

Because of the 'a^2', (1) is quite different from the inequations we have dicussed. So a sensible thing to do is to go back to Definition 3-7-1. This definition tells us that (1) is equivalent to

$$a^2 + 4 = 5(2 - a) + p \text{ (where p is positive)}$$

this means $a^2 + 4 - 5(2 - a)$ is positive

or

(2) $a^2 + 5a - 6$ is positive

As it stands, (2) may not seem to be much of an improvement over (1). But your experience with this kind of equation may suggest factoring. If we do this, we find that (2) is equivalent to:

(3) $(a + 6)(a - 1)$ is positive

This raises the question "When is a product positive?" You probably know the answer to this question. If you do, you see that (3) is equivalent to:

$a + 6$ and $a - 1$ are both positive

or

$a + 6$ and $a - 1$ are both negative

Now we can go back to inequations of a kind we know how to handle:

(4) $a \geq -6$ and $a > 1$

or

($a + 6 < 0$ and $a - 1 < 0$)

By our transformation principles, (4) is equivalent to:

(5) ($a > -6$ and $a < 1$)

or

($a < -6$ and $a < 1$)

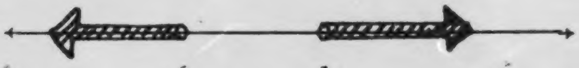
Finally, since $1 > -6$ it follows that ($a > -6$ and $a > 1$) if and only if $a > 1$ and that ($a < -6$ and $a < 1$) if and only if $a < -6$. Hence, (5) is equivalent to:

(6) $a > 1$ or $a < -6$

So the solution set of (1) is the union of the solution sets of '$a > 1$' and '$a < -6$'. We shall be satisfied with saying that the solution set of (1) is

$$\{a \mid a > 1 \text{ or } a < -6\}$$

A graph of the solution set of (1) is:



The preceding work suggests the following generalization.

Theorem 3-7-15: The Factoring Transformation Principle for Inequations. (FTPI)

$\forall x \forall y,\ xy > 0 \leftrightarrow ((x > 0 \text{ and } y > 0) \text{ or } (x < 0 \text{ and } y < 0))$

$\forall x \forall y,\ xy < 0 \leftrightarrow ((x > 0 \text{ and } y < 0) \text{ or } (x < 0 \text{ and } y > 0))$

There are analogous theorems which can be obtained by replacing, everywhere, '$>$' by '$\geq$' and '$<$' by '$\leq$'.

We can now carry out the solution of (1) as follows:

Example 3-7-16:

$$a^2 + 4 > 5(2 - a)$$

$$a^2 + 4 > 10 - 5a$$

$$a^2 + 5a - 6 > 0$$

$$(a + 6)(a - 1) > 0$$

$$(a + 6 > 0 \text{ and } a - 1 > 0) \text{ or } (a + 6 < 0 \text{ and } a - 1 < 0)$$

$$(a > -6 \text{ and } a > 1) \text{ or } (a < -6 \text{ and } a < 1)$$

$$a > 1 \text{ or } a < -6$$

The solution set is $\{u \mid u > 1 \text{ or } u < -6\}$

Example 3-7-17: Find the solution set of $2a^2 < 5a + 3$

1. $2a^2 \leq 5a + 3$	1. Assumption
2. $2a^2 - 5a - 3 \leq 0$	2. ATPI & simplification
3. $(2a + 1)(a - 3) \leq 0$	3. Factoring
4. $(2a + 1 \leq 0 \text{ and } a - 3 \geq 0)$ or $(2a + 1 \geq 0 \text{ and } a - 3 \leq 0)$	4. Theorem 3-7-15
5. $(a \leq -\frac{1}{2} \text{ and } a \geq 3)$ or $(a \geq -\frac{1}{2} \text{ and } a \leq 3)$	5. ATPI and MTPI

The solution set is $\emptyset \cup \{a \mid -\frac{1}{2} \leq a \leq 3\} = \{a \mid -\frac{1}{2} \leq a \leq 3\}$

Exercise 3-7-18:

Find the solution set for each of the following inequations and draw a graph of this solution set:

1. $(x + 4)(x - 2) > 0$
2. $(x + 7)(x - 3) < 0$
3. $x^2 - 2x - 8 > 0$
4. $a^2 + 2(a - 2) < a + 8$
5. $2a^2 - 9a - 35 > 0$

Many times the singulary operation "Absolute Value" appears in inequations.

Definition 3-7-19:

$$\forall a \ |a| = a \text{ if and only if } a \geq 0$$

$$|a| = -a \text{ if and only if } a < 0$$

The following theorem follows directly from the above definition.

Theorem 3-7-20:

$$\forall x \forall y \geq 0 \quad |x| = y \leftrightarrow x = y \text{ or } -x = y$$

We are now ready to solve some equations involving the absolute value operation.

Example 3-7-21: Find the solution set of $|3x - 2| = 2$

$$|3x - 2| = 2$$

$$3x - 2 = 2 \text{ or } -(3x - 2) = 2$$

$$x = \frac{4}{3} \text{ or } x = 0$$

$$\text{Solution set is } \{\tfrac{4}{3}, 0\}$$

Example 3-7-22: Find the solution set of $|2x - 5| = 2 + x$

$$|2x - 5| = 2 + x \qquad (2 + x \geq 0)$$

$$(2x - 5 = 2 + x \text{ or } -(2x - 5) = 2 + x) \text{ and } x \geq -2$$

$$(x = 7 \text{ or } -3x = -3) \text{ and } x \geq -2$$

$$(x = 7 \text{ or } x = 1) \text{ and } x \geq -2$$

$$\text{Solution set } (7, 1)$$

Example 3-7-23: Find the solution set of $|3x - 2| = x - 1$

$$|3x - 2| = x - 1$$

$$(3x - 2 = x - 1 \text{ or } -(3x - 2) = x - 1) \text{ and } x - 1 \geq 0$$

$$(2x = 1 \text{ or } -3x + 2 = x - 1) \text{ and } x \geq 1$$

$$\left(x = \frac{1}{2} \text{ or } -4x = -3\right) \text{ and } x \geq 1$$

$$\left(x = \frac{1}{2} \text{ or } x = \frac{3}{4}\right) \text{ and } x \geq 1$$

Solution set $= \emptyset$

Exercise 3-7-24: Find the solution set of each of the following.

1. $|3x| = 12$
2. $|2x + 1| = 7$
3. $|2x| = 2x$
4. $|2x| = 4x$
5. $|x - 1| = 2 - x$
6. $|6 - 3x| = 2x + 4$
7. $|x + 1| = |x + 2|$
8. $|x + 1| = |x| + 1$

Classify the following generalizations as true or false.
If false, show a counter example.

9. $\forall x\ |-x| = |x|$
10. $\forall x\ |x| \geq 0$
11. $\forall x \forall y\ |x| + |y| = |x + y|$
12. $\forall x \forall y\ |x| \cdot |y| = |x \cdot y|$
13. $\forall x \forall y\ \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
14. $\forall x\ |x^2| = |x|^2 = x^2$
15. $\forall x \forall y\ |x - y| = |x| - |y|$

Here is a proof of generalization (1), $\forall x\ |-x| = |x|$

For $x \geq 0$, $-x \leq 0$ and, by definition, $|-x| = --x$

But $\forall x\ --x = x$. Therefore $|-x| = x$

By definition, for $x \geq 0$, $|x| = x$

Therefore for $x \geq 0$, $|-x| = x = |x|$

For $x \leq 0$, $-x \geq 0$ and by definition, $|-x| = -x$

By definition, for $x \leq 0$, $|x| = -x$

Thus, for $x \leq 0$ $|-x| = -x = |x|$

Since for $x \geq 0$, $|-x| = |x|$

and

for $x \leq 0$ $|-x| = |x|$

it follows that $\forall x \ |-x| = |x|$

Problems Set 3-7-25:

1. Write proofs of the other true generalizations as directed by your teacher.

We will now investigate inequations involving the absolute value operation.

Exercise 3-7-26:

Find the solution sets of the following inequations and graph them on a number line.

1. $|x| < 3$
2. $|x| > 3$
3. $|x - 2| < 4$
4. $|x - 2| > 4$

The solutions to the problems in the above Exercise (3-7-26) suggest the following theorems.

Theorem 3-7-27: $\forall x \forall y \ |x| < y \rightarrow -y < x < y$

Theorem 3-7-28: $\forall x \forall y \ |x| > y \rightarrow x > y$ or $x < -y$

These theorems are useful in solving inequations containing absolute value expressions.

Example 3-7-29: Find the solution set of $|x + 2| < 7 - x$

$$|x + 2| < 7 - x$$

$$-(7 - x) < x + 2 < 7 - x$$

$$-7 + x < x + 2 \text{ and } x + 2 < 7 - x$$

$$-7 < 2 \text{ and } 2x < 5$$

$$-7 < 2 \text{ and } x < \frac{5}{2}$$

Since each real number satisfies $-7 < 2$ and those less than $\frac{5}{2}$ satisfy $x < \frac{5}{2}$ the solution set is $\{x | x < \frac{5}{2}\}$

Example 3-7-30: Find the solution set to $|x - 3| > 4$

$$|x - 3| > 4$$

$$x - 3 > 4 \text{ or } x - 3 < -4$$

$$x > 7 \text{ or } x < -1$$

The solution set is $\{x|x > 7\} \cup \{x|x < -1\}$

or

$$\{x|x < -1 \text{ or } x > 7\}$$

Exercise 3-7-31: Find the solution sets of the following inequations:

1. $|3x| < 2$
2. $|x - \frac{1}{2}| \leq \frac{3}{2}$
3. $|2x| > 3$
4. $|x - 1| < 3$
5. $|\frac{x}{3}| < 2$
6. $|x - 1| > 8$
7. $|2 - x| > 1$
8. $\left|\frac{2x^2 - x}{x}\right| \leq 4$
9. $|2x - 1| \geq 3$

The converse of Theorem 3-7-27 is also a thereom.

Theorem 3-7-32: $\forall x \forall y \quad -y < x < y \rightarrow |x| < y$

This theorem provides a means of translating conjunctions of inequations into absolute value sentences.

Example 3-7-33: Find an absolute value expression equivalent to $-7 < x < 7$

Solution: $|x| < 7$

Example 3-7-34: Find an absolute value expression equivalent to $-2 < x < 8$.

We need to find some number 'a' such that,

(1) $-2 + a < x + a < 8 + a$ and (2) $-2 + a = -(8 + a)$

(2) $-2 + a = -(8 + a)$

$$-2 + a = -8 - a$$

$$2a = -6$$

$$a = -3$$

Substituting -3 into (1) $-2 - 3 < x - 3 < 8 - 3$

$$-5 < x - 3 < 5$$

$$|x - 3| < 5$$

Exercise 3-7-35: Replace each of the following inequalities with a statement using an inequality and the absolute value sign.

1. $1 < x < 3$
2. $-2 \leq x \leq 4$
3. $1 < 2x + 1 < 6$
4. $3 < 1 - 2x < 5$
5. $-3 < 2x - 1 < \frac{1}{2}$
6. $-3 < 1 - 2x < \frac{1}{2}$

Problem Set 3-7-36:

Solve the following inequations.

1. $|x - 1| < 3$
2. $||x| - 3| < 2$
3. $|x - 1| \leq |x| + 1$
4. $|\frac{1}{x} - 2| > 2$
5. $|x - 3| \geq |x - 2|$
6. $|x^2 - 7| < 2$
7. $|x^2 - 5x - 3| < 3$
8. $|x(x - 7)| \geq x$

Problem Set 3-7-37: Here are some more generalizations containing absolute value expressions. Classify each as true or false.

1. $\forall a \forall b \quad |a - b| \leq |a| + |b|$
2. $\forall a \forall b \quad |a - b| \leq |a| - |b|$
3. $\forall a \forall b \quad |a + b| \leq |a| + |b|$
4. $\forall a \forall b \quad |a| - |b| \leq |a - b|$
5. $\forall a \forall b \quad |a| - |b| \leq |a + b|$
6. $\forall a \forall b \quad |a - b| \leq |a + b|$
7. $\forall a \forall b \quad |a| < |b| \rightarrow |a|^2 < |b|^2$

3-8 Problems for Application

Example 3-8-1

If you increase a certain number by 17, you get the same result as if you had subtracted $\frac{1}{2}$ the number from 5. What is this number?

Solution. We know that, for each number x, if you increase x by 17, you get $x + 17$, and if you subtract $\frac{1}{2}x$ from 5, you get $5 - \frac{1}{2}x$. So, we want to find a number x such that:

(1) $x + 17 = 5 - \frac{1}{2}x$

If there is a number which meets the conditions of this problem, it is a root of (1). So, we solve (1):

$$x + 17 = 5 - \frac{1}{2}x$$

$$2x + 34 = 10 - x$$

$$3x = -24$$

$$x = -8$$

Instead of checking to see whether the root of (1) is -8, we check to see whether -8 fits the conditions of the problem (Why do this?)

Check.

-8 increased by 17 is 9;

the difference of $\frac{1}{2} \cdot -8$ from 5 is 9.✓

Answer. The number in question is -8.

Example 3-8-2

John usually takes 40 minutes to ride his bicycle from home to school. When he is pressed for time, he can increase his average speed by 6 miles per hour and save 16 minutes. How far does John live from school?

Solution. If John lives x miles from school and it takes him $\frac{2}{3}$ of an hour (40 minutes) at his usual rate to make the trip from home to school, his usual rate is $\frac{x}{\frac{2}{3}}$ (or: $\frac{3x}{2}$) miles per hour. If he increases this rate by 6 miles per hour, the new rate is $\frac{3x}{2} + 6$ miles per hour. At this new rate the time required for the trip is 16 minutes less than the usual time of 40 minutes. That is, the new time is 24 minutes, or $\frac{2}{5}$ of an hour. So, we are looking for a number x of arithmetic such that

$$(\frac{3x}{2} + 6)\frac{2}{5} = x.$$

Let's solve this equation.

$$\left(\frac{3x}{2} + 6\right)\frac{2}{5} = x$$

$$\frac{3x + 12}{2} \cdot \frac{2}{5} = x$$

$$\frac{3x + 12}{5} = x$$

$$\frac{3x + 12}{5} \cdot 5 = x5$$

$$3x + 12 = 5x$$

$$12 = 2x$$

$$6 = x$$

Check. If John lives 6 miles from school and it takes 40 minutes to make the trip from home to school, his usual rate is $6 \div \frac{2}{3}$ miles per hour; that is, 9 miles per hour. Now, if he increases his usual rate by 6 miles per hour, his new rate will be 15 miles per hour. Will this new rate make it possible for him to get to school in 16 minutes less time (i.e., in 24 minutes), as the problem stated? Well, if he lives 6 miles from school, and travels at a rate of 15 miles per hour, it will take him $6 \div 15$ hours to get there. $6 \div 15$ is $\frac{2}{5}$ hours, or 24 minutes--which is 16 minutes less than 40 minutes!

Answer. John lives 6 miles from school.

Example 3-8-3

How many quarts of a 30% alcohol solution should be added to 8 quarts of a 40% alcohol solution to make a new solution which is 38% alcohol?

Solution. Suppose you add x quarts of the 30% alcohol solution to the 8 quarts of the 40% alcohol solution. Since what you add contains .3x quarts of alcohol, and since the original 8 quarts of solution contain .4(8) quarts of alcohol, the new solution contains (.3x + .4(8)) quarts of alcohol. But, the new solution contains a total of (x + 8) quarts of liquid. So, we are looking for a number x such that

$$.3x + .4(8) = .38(x + 8).$$

We solve this equation.

$$100(.3x + .4(8)) = (.38(x + 8))100$$

$$30x + 320 = 38x + 304$$

$$16 = 8x$$

$$x = 2$$

Check. 2 quarts of a 30% alcohol solution contain .6 quarts of alcohol.

8 quarts of a 40% alcohol solution contain 3.2 quarts of alcohol.

So, the new mixture of 10 quarts of solution contains 3.8 quarts of alcohol, and 3.8 is 38% of 10.

Answer. 2 quarts of a 30% alcohol solution should be added.

Example 3-8-4

Mr. Alders invests a total of $3800 in two enterprises, one giving an income of 3% and the other an income of 5%. If the total income from these investments is $166, how much is invested in each enterprise?

Solution. Suppose he invests x dollars at 3%. Then he invests (3800 - x) dollars at 5%. The income from these investments is

$$.03x + .05(3800 - x) \text{ dollars}$$

So, we are looking for a number x such that

$$.03x + .05(3800 - x) = 166.$$

(Finish the solution and check.)

Example 3-8-5

If Albert can mow a lawn in 2 hours and Bill can mow this lawn in 3 hours, how long will it take them to mow the lawn if they work together?

Solution. Method I

You know that Albert will mow $\frac{1}{2}$ of the lawn in one hour, and Bill will mow $\frac{1}{3}$ of it in one hour, if they work at steady rates. Suppose they work together for x hours. Then, together, they would mow $\frac{1}{x}$ part of the lawn in 1 hour. So, we are looking for a number x such that

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{x}$$

(Solve this equation and check.)

Method II

Suppose it takes x hours to mow the lawn if both boys work together. Then Albert would mow $\frac{x}{2}$ part of the lawn during this time, and Bill would mow $\frac{x}{3}$ part of it at the same time, and they would be finished. So, we need to find a number such that

$$\frac{x}{2} + \frac{x}{3} = 1$$

(Solve this equation and check.)

Problem Set 3-8-6

Solve each of the following problems. Check your results carefully.

1. One pint of an alcohol solution contains 15% alcohol. How much pure alcohol (100% alcohol solution) must be added to make a solution which contains 35% alcohol?
2. Jim picked a number, tripled it, added 4 to the result, divided the sum by 8, and got 5. What number did he pick?

3. Edward is two years older than Charles. Eleven years ago Edward was twice as old as Charles. How old is each boy now?

4. A business man has 7 minutes to catch a train at a station which is 8 miles from his home. His taxi covers half of this distance traveling at an average speed of 30 miles per hour. What should be the average speed of the taxi during the second half of the trip to enable the man to catch the train?

5. A confectioner is making a mixture of almonds and cashews. The cashews are worth $.90 a pound and the almonds are worth $.75 a pound. How many pounds of each kind of nut should be used to make 30 pounds of a mixture worth $.81 per pound?

6. Two boys start around a 1300-foot track, running in opposite directions. If one boy runs 6 feet more per second than the other, and they meet in 24 seconds, what is the rate of the faster boy?

7. Two cyclists start at the same time and from the same place and travel in opposite directions. In twenty minutes they are 11 miles apart. The faster cyclist travels at an average speed which is 3 miles per hour more than the average speed of the slower cyclist. What is the average speed of each cyclist?

8. A freight train and a passenger train on parallel tracks are 7 miles apart at 1:00 p.m. and are traveling in opposite directions. The passenger train's average speed is 35 miles per hour more

than the average speed of the freight train. If they maintain their average speeds and are 45 miles apart at 1:24 p.m., what is the average speed of the freight train?

9. Bill can mow a lawn in 35 minutes and his brother can do the same job in 40 minutes. If they were to work together, how long would they take to mow the lawn?

10. A tank has two inlet pipes. One pipe by itself can fill the tank in 17 minutes; the other pipe by itself can fill the tank in 21 minutes. How long will it take to fill the tank if both pipes are opened?

11. There are 783 pupils in Zabranchburg High School. If the ratio of girls to boys is 5 to 4, how many boys are there in the school? (If there are 4x boys there are 5x girls.)

12. A man has a total of $3000 earning interest, some at 5% and the remainder at 6%. The amount of annual interest on both investments is $155. How much is invested at each rate?

13. A man who can row 5 miles an hour in still water rows up a stream for 3 hours and then rows back to his starting point in 2 hours. At what rate does the stream flow?

14. Howard can read at a rate which is 4 pages an hour more than one and a half times Paul's rate. If Howard were to decrease his rate by 50% and if Paul were to increase his by 20%, each would read a book of 397 pages in 12 hours. What is Paul's usual reading rate?

15. A man has \$3.50 in dimes and quarters. He has 17 coins in all. How many coins of each denomination does he have?

16. Divide \$155 among A, B, C, and D so that A and B together receive \$40, C receives twice as much as A, and D receives three times as much as B.

17. If -6 is added to half a certain number, the result is 15. What is the number?

18. Jack wants a sweater that costs \$.15 more than 3 times the amount of money he now has. If the sweater costs \$4.50, how much money does Jack have now?

19. Herbert is walking up a long flight of steps. He climbs 6 less than half the total number, then he climbs 4 more than a third of the number remaining. He rests for a while, and then climbs 3 less than a fourth of what still remains. There are 48 steps left to reach the top. How many steps are there in the flight?

20. Noodles, Bismark, and Clem are three dachshund puppies. Noodles is one hour more than half as old as Bismark, and 3 hours older than Clem. Four hours ago Bismark's age was $4\frac{2}{3}$ Clem's age. How old is each puppy?

21. Two grades of coffee were accidentally mixed, and thus produced a new grade. To meet the cost of advertising this new grade, the coffee distributor had to pay 7% of the gross income he had expected to receive from the sale of the original two grades (11.7 tons for \$17,550 and 8.3 tons for \$14,940). How much (to the nearest cent) per pound must he charge for the new grade to meet the advertising costs, and to give him his originally expected income?

22. How many pounds of coffee at 75 cents per pound should be mixes with 337 pounds of coffee at 90 cents per pound to produce 1000 pounds of mixture worth 80 cents per pound?

23. How many pounds of coffee at 75 cents per pound should be mixed with 337 pounds at 90 cents per pound to produce a mixture worth (a) 50 cents per pound, (b) 80 cents per pound, (c) $1.00 per pound?

24. Andrew has twice as much money as Scott. If Andrew were to lend Scott a quarter then both boys would have the same amount of money. How much money does each boy have?

25. (a) Two bees working together, can gather nectar from 100 hollyhock blossoms in 30 minutes. Assuming that each bee works the standard eight-hour day, five days a week, how many blossoms do these bees gather nectar from in a summer season of fifteen weeks?

    (b) In working on a batch of 100 blossoms, one of the bees stops after 18 minutes (just to smell the flowers), and it takes the other bee 20 minutes to finish the batch. How long would it take the diligent bee to gather nectar from 100 blossoms if she worked all by herself?

3-9 Miscellaneous Review Problems

Review Problems Set: 3-9-1

Solve each of the following sentences:

1. $7(x - 2) - 2(3 + x) = 0$
2. $3(2a - 9) = 5(10 - a)$
3. $20x + 10(2x) + 5(2x + 6) + (5x + 6) = 806$
4. $-4x + 2(5x + 1) - 5 = 6 - 3(2x-1)$
5. $-14(7c + 10) = 7c(4c - 7) - 4c(7c - 1)$
6. $\frac{1}{2} y + \frac{1}{10y} + \frac{1}{5} y = y - 6$
7. $2(100 - p) + 1.6p = 1.75(100)$
8. $\frac{2x + 3}{2} = -5$
9. $\frac{3x}{5} = \frac{19}{5} + \frac{5x}{2}$
10. $\frac{4m}{3} + \frac{3m}{5} = \frac{7m}{2}$
11. $\frac{a - 1}{3} = 2$
12. $c - 0.70c = 67.5$
13. $\frac{1}{2}(970 - a) + \frac{3}{4} a = 550$
14. $\frac{a + 2}{3} + \frac{1}{2} (a + 3) = 3$
15. $\frac{1}{3}(5 - 7x) + \frac{1}{2}(4x + 7) = \frac{1}{2}(3x + 31)$
16. $4 - \frac{3b - 5}{b} = \frac{5}{b}$
17. $\frac{5}{2y} + \frac{1}{3y} = \frac{1}{y} + 2$
18. $\frac{7}{2b} - 6 = 9 - \frac{5}{4b}$
19. $\frac{9}{b + 6} - \frac{8}{4 - b} = \frac{12}{b - 4}$
20. $\frac{180}{\frac{3}{2} x} = \frac{180}{x} - 2$
21. $\frac{5}{2} = \frac{m + 7}{m + 3}$
22. $\frac{44}{11 - x} = \frac{4x}{11 - x} - \frac{19}{3}$
23. $\frac{2}{x + 4} = \frac{1}{x - 3}$
24. $x - \frac{16}{x - 4} = \frac{4x}{4 - x}$
25. $8(x - 7) = (3x + 2)(x - 7)$
26. $2x - 3 < 11$
27. $5z - 4 < 2z + 5$
28. $2(c + 1) - 3 \leq 8 - c$
29. $-1 < 2x + 1 < 1$
30. $|x| + 4 = 0$
31. $-5 \leq 3x + 1 \leq 10$
32. $|m - 5| = 6$
33. $-|1 - 2x| = 5$
34. $13 - |6 - 3x| = 4$
35. $18 - 2 |y + 3| = 12$
36. $6 < 3x + 2 < 10$
37. $-2 < \frac{2w + 3}{5} < 2$
38. $2 < \frac{5 - 3y}{2} < 15$

39. $-3 \leq 4 - 2a < 3$

40. $-1 \leq \frac{4 - 3x}{-2} \leq 1$

41. $|x + 3| < 5$

42. $|3y - 2| \leq y + 2$

43. $|4 - 2m| \geq 6$

44. $|2 - p| > 4 + 2p$

45. $1 \leq |x + 2| \leq 3$

Solve each equation for the variable indicated.

46. $A = \frac{1}{2} BH$; B

47. $X = Y + YZ$; Z

48. $L = A + (N - 1)D$; D

49. $A = \frac{1}{2} H (B' + B'')$; H

50. $R = \frac{E}{I}$; I

RELATIONS AND FUNCTIONS

4-1 Introduction.

Up to this point in the course, we have studied the properties of the real numbers and developed a structure known as the algebra of the reals. Now we concern ourselves with an interesting notion, the relation of one real number to another. Real numbers are related to each other in many different ways, but one class of relations, known as functions, is so important in mathematics that it will receive special attention in this chapter, and will in fact, become the unifying theme for the rest of the course.

4-2 Relations.

Our everyday activities bring us into contact with many situations that have the property of relating two elements. For example, on a test, to each student there is assigned a grade; when you buy gasoline, to each quantity purchased there is associated a total cost; in a baseball game for each inning there is a total score; and for each husband there is a wife. The list continues without end. Even in the realm of nonsense we find this condition: A witch doctor has given the following rule to determine a person's blood pressure: blood pressure = one-half a person's age plus 110.

In each of these examples there are pairs of elements: (student, grade), (number of gallons, cost), (inning, total score), (husband, wife), and (person's age, blood pressure). Problems such as these involve the use of two or more variables. Sometimes the order in which two quantities are listed is not important. For example, the set of natural numbers less than three could be described as $\{1,2\}$, or as $\{2,1\}$ and each is as correct as the other. Often, however, the order of listing is important. Consider, for example, the following inning-total score ordered pairs. We might have pairs of numbers such as $\{(1,1), (2,1), (3,3), (4,4), (5,7), (6,7)\}$. These pairs are meaningless unless we can tell which number of the pair is the inning and which is the total score. Certainly then, the order in which these numbers are listed is important. If it is specified that the order of the elements of these pairs should be inning first and total score second, then the set conveys meaningful information.

When we consider a pair of elements in which the order of listing is important, the pair is called an ORDERED PAIR and is written (a,b). The ordered pair (b,a) is not the same as the ordered pair (a,b). Each of the components in the pair is called an element of the ordered pair, with the first one listed being referred to as the FIRST ELEMENT and the second one listed called the SECOND ELEMENT.

Two ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$, that is, if and only if corresponding elements are equal. Thus $(2/3, 3/4) = (4/6, 6/8)$ and $(3 + 2, 1) = (5, (4-3))$. However, $(2,3) \neq (3,2)$ and $(1/3, 4/5) \neq (.5, .3)$.

Each of the situations (relations) described previously in this section could be represented by a set of ordered pairs. To aid in our discussion of sets of ordered pairs, we make the following definition.

Definition 4-2-1 Relation

A RELATION is a set of ordered pairs.

In order to be able to refer to the set of first elements and the set of second elements, we define the set of all first elements as the DOMAIN of the relation and the set of all second elements as the RANGE of the relation. Here are two examples:

Example: 4-2-2 The set of ordered pairs:

$K = \{(1,2)\ (3,4)\ (5,6)\ (7,8)\ (9,10)\}$

is a relation. The domain of this relation is the set of all first elements:

$\{1,3,5,7,9\}$

The range of this relation is the set of all second elements:

$\{2,4,6,8,10\}$

Example: 4-2-3.

In the relation:

$R = \{(-3,4)(6,2)(5,2)(2,6)(7,4)(6,9)\}$

the domain is $\{-3,6,5,2,7\}$ and the range is $\{4,2,6,9\}$.

Exercise Set - 4-2-4

1. Given the following relation

   $C = \{(2,3)(10,6)(5,3)(9,6)(6,6)(3,6)\}$

   a. State the domain of this relation.
   b. State the range of this relation.
   c. State the set of elements in the intersection of the domain and range.

2. The set of ordered pairs:

   $W = \{(1,3)(1,5)(1,7)\}$ is a relation.

   a. What is the domain of this relation?
   b. What is the range of this relation?
   c. State the set of elements resulting from the union of the domain and range and the intersection of the domain and range.

Exercise Set - 4-2-4 (cont.)

3. The set of ordered pairs:

   $Z = \{(a,b)(c,d)(m,n)(y,z)\}$ is a relation.

   a. State the domain of this relation.
   b. State the range of this relation.

4. Using the definition of equality of ordered pairs, solve for x and y in each of the following exercises.

   a. $(2x - 3x, 4y + 9y) = (1,3)$
   b. $(3x + 4x, -2y + 9) = (5.5,4.5)$
   c. $(-7x + 2, -5 + 3y) = (2x - 4, 5)$

Sometimes a situation determines which element of the range is to be paired with a particular element of the domain. As in the gasoline purchase situation, the cost per gallon determines the total cost associated with each number of gallons purchased. For instance, 10 gallons purchased at 39.9 cents per gallon would be represented by (10,3.99). However, the definition of a relation does not require that there be a "way" by which elements are to be paired. For instance, we might arbitrarily pick ordered pairs of real numbers to form the set

$B = \{(1,-1)(\frac{1}{2},0)(2,7)(2,3)\}$ which would be a relation.

Most of the relations which we will study in this course will be infinite sets of ordered pairs of real numbers. The numbers making up an ordered pair are associated with each other by some rule or relationship in many instances. This situation is best described by a technique called "set builder notation".

Definition 4-2-5 Set Builder Notation

$\{(x,y) \mid y \,\theta\, x,\ x \in R,\ y \in R\}$ is defined as the set of all ordered pairs of real numbers (x,y) such that the second element of each pair is related to its corresponding first element by the relationship θ.

In the set builder notation definition, the expression, $y \,\theta\, x$, represents one or more equations or inequations in the variables x and y. They are referred to as the set selectors and are used to determine which elements (x,y), $x \in R$, $y \in R$, belong to the relation.

Example - 4-2-6

Consider the following relation:

$$\{(x,y) \mid y = 2x + 1,\ x,\ y \in R\}$$

This relation is the set of all ordered pairs of real numbers which satisfy the equation $y = 2x + 1$. In this exercise the relationship $y \,\theta\, x$ is $y = 2x + 1$. The equation is referred to as the set selector because it must be used to determine which elements belong to the relation as shown below.

The ordered pair $(-13,246)$ is not an element of $\{(x,y) \mid y = 2x + 1,\ x \in R,\ y \in R\}$ because we get the false statement, $246 = 2(-13) + 1$, when $(-13,246)$ is substituted in the set selector equation $y = 2x + 1$.

In a like manner we know that the ordered pair $(326,653)$ is an element of $\{(x,y) \mid y = 2x + 1,\ x \in R,\ y \in R\}$ because we get the true statement $653 = 2(326) + 1$ when $(326,653)$ is substituted into the set selector equation.

For some restrictions of the replacement set for x, in the above relation, a finite set of ordered pairs will result. For instance, restrict x so that $0 \leq x \leq 10$, x is an integer and then the relation would consist of these 11 ordered pairs: (0,1) (1,3) (2,5) (3,7) (4,9) (5,11) (6,13) (7,15) (8,17) (9,19) (10,21). Using set-builder notation we can describe this set as:

$$\{(x,y) \mid y = 2x + 1,\ x \in I,\ 0 \leq x \leq 10\}$$

Exercise - 4-2-7

1. Using set-builder notation, describe the set that would relate blood pressure to age using the witch doctor's method. (See page 4-1).
2. Make a flow chart to illustrate the algorithm for forming the set of ordered pairs in problem 1.
3. Use the set builder notation to describe each of the following relations.

   a. $\{(-3,9)\ (-2,4)\ (-1,1)\ (0,0)\ (1,1)\ (2,4)\ (3,9)\}$

   b. $\{(1,2)\ (2,4)\ (3,6)\ (4,8)\ (5,10)\ (6,12)\ (7,14)\}$

   c. $\{(-3,-4)\ (-2,2)\ (-1,0)\ (0,2)\ (1,4)\ (2,6)\ (3,8)\}$

4. The birth dates of the following boys are: Sam, December 31, 1954; Henry, January 1, 1954; Oliver, June 15, 1955; Robert, April 15, 1955; and Mike, January 1, 1955. If "age means a person's age in years" at his last birthday, list the set of ordered pairs in the following relation:

   a. $\{(x,y) \mid y \,\theta\, x$, where θ is the relationship "is older than", and the ages are as described above.$\}$

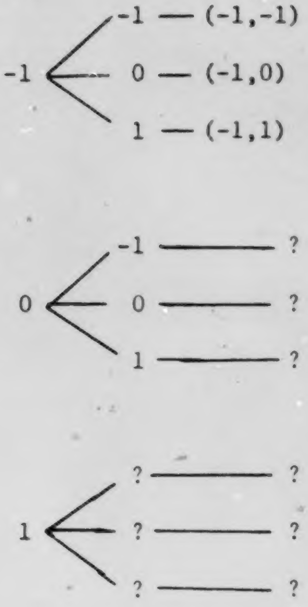
   b. $\{(x,y) \mid y \,\theta\, x$, where θ is the relationship "is the same age as", and the ages are as described above.$\}$

Another way of generating a relation is to pair every element of a set A with each element of a set B, (A and B might be the same set). The set of ordered pairs resulting from this process is called a "Cartesian product" and is denoted by A X B. The X does not indicate multiplication but is read "A cross B." If B = {0, 1, 2, 3}, and A = {a, b, c}, then A X B = {(a, 0), (a, 1), (a,2), (a,3), (b,0), (b,1), (b,2), (b,3), (c,0), (c,1), (c,2), (c,3)}. Set A is the domain and Set B the range of the relation A X B.

If we take the set of real numbers R, and form the set R X R, a relation consisting of an infinite set of ordered pairs would result. Most of our study of algebra deals with subsets of R X R. For example, $\{(x,y) \mid y = \sqrt{x}, x \geq 0$, x an integer$\}$ is a subset of R X R.

Problem Set 4-2-8

1. Let U = {-1, 0, 1}. If we wish to form all ordered pairs whose members belong to U, one way of doing it is illustrated, in part, below. You complete the "tree."



2 Given set A = {2, -7, 9, -9, -2, 7}

Form the relation A X A.

Problem Set 4-2-8 (cont.)

3. List the ordered pairs in each subset of A X A described by the following rules.

   a. The second element is greater than the first.
   b. The first element is greater than or equal to the second.

Given set B = { 16, 8, 4, 2 }

4. Form the relation B X B

5. List the ordered pairs in each subset of B X B described by the following rules:

   a. The second element is a multiple of the first.
   b. The first element is the square root of the second.
   c. The second element is one-fourth of the first.

Given the set C = $\{\pi, 22/7, 3.\dot{1}4159, 3\frac{10}{71},\}$

6. Form the relation C X C

7. List the ordered pairs in each subset of C X C described by the following rules:

   a. The first element is to the right of the second when plotted on the number line.
   b. The first element is to the left of the second when plotted on the number line.
   c. Explain how the two sets of ordered pairs in a and b are related.

8. Write computer programs to generate:

   a. $\{(x,y) \mid y = \sqrt{x}, 0 \le x \le 10$, x an integer$\}$

   b. $\{(a,p) \mid p = \frac{1}{2}a + 110\}$ a = person's age, p = blood pressure.

9. Given A = $\{x \mid -3 \le x \le 3$, x an integer$\}$

   a. Write a computer program to generate: A X A

   b. Write a program to generate the following sets (use square root function):

      R = $\{(x,y) \mid y = \sqrt{x}, -3 \le x \le 3$, x an integer$\}$

      S = $\{(x,y) \mid y = x, -3 \le x \le 3$, x an integer$\}$

   c. Which of the relations in part b are subsets of A X A?

4-3 Graphing Relations.

In this section you will learn how to associate ordered pairs of real numbers with points in a plane by establishing a frame of reference, a standard unit of measure, and certain conventions regarding positive and negative directions. You will also learn how to graph a relation, and write a program to sort numbers.

To establish our frame of reference, we choose two perpendicular lines in the plane, called the coordinate axes, and designate their intersection as the origin. A real-number scale with zero at the origin is established on each of the axes. The unit of length is the same on both axes. Figure 4-3-1 illustrates the coordinate axes as they are conventionally drawn.

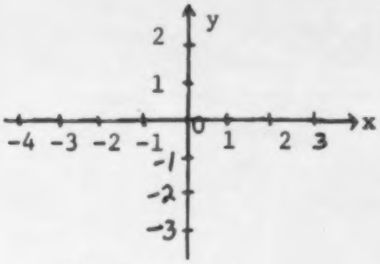


Figure 4-3-1

Note that the horizontal axis is labeled as the x-axis and the vertical axis as the y axis. Arrowheads are used to indicate the positive directions for the number scales on the x and y axes. Two lines thusly oriented in a plane establish a system known as the Cartesian (or rectangular) coordinate system.

The Cartesian coordinate system establishes a one-to-one correspondence between R X R and the set of points in the plane. To each arbitrary point in the plane there is associated a unique element of R X R and to each arbitrary element of R X R there is associated a unique point in the plane.

Let P be an arbitary point in the Cartesian plane, as shown in Figure 4-3-2. If a line is drawn through P perpendicular to the x-axis, this line will intersect the x-axis in a unique point corresponding to the real number a on the number scale of the x-axis. Similarly, a line through P perpendicular to the y-axis intersects the y-axis in a unique point corresponding to the real number b on the number scale of the y-axis.

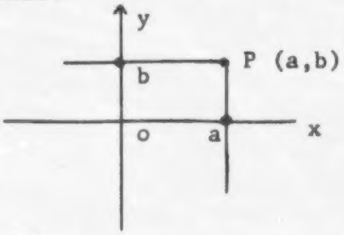


Figure 4-3-2

We associate with the point P the ordered pair of real numbers (a,b) which, of course, is an element of R X R. It can be seen that any point P in the plane can be represented by a unique ordered pair of real numbers (a,b). The components a and b of the ordered pair of real numbers (a,b) are the coordinates of the point P. The number a is called the x-coordinate or the abscissa of the point P and b is called the y-coordinate or the ordinate of the point P.

The coordinate axes separate the plane into four distinct regions, called quadrants, as shown in Figure4-3-3. Both coordinates of any point located in the first quadrant are positive. This is indicated in Figure 4-3-3 by the symbol (+,+). The other quadrants and the signs of the coordinates of points in these quadrants are also indicated in the diagram.

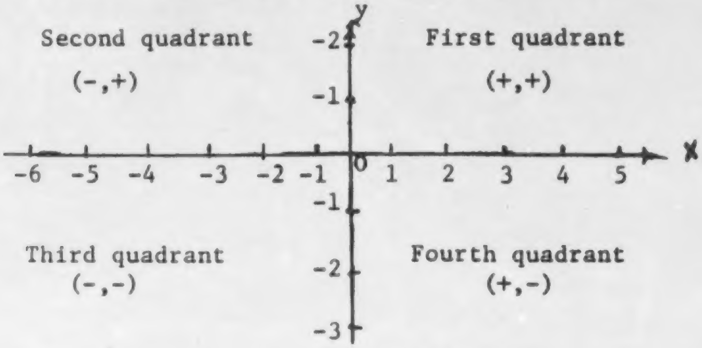


Figure 4-3-3

Each of the points on the x-axis to the right of the origin, has a positive abscissa and an ordinate of zero. Each of the points to the left of the origin on the axis has a negative abscissa and an ordinate of zero. Similarly, each point on the y-axis above the origin, has a positive ordinate and a zero abscissa. Each point on the y-axis, below the origin, has a negative ordinate and a zero abscissa.

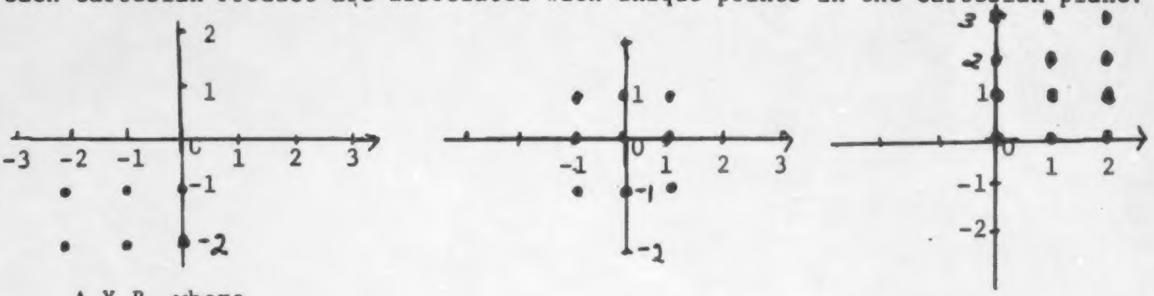
By a similar procedure it can be shown that any ordered pair of real numbers corresponds to an unique point in the plane. Given any ordered pair of real numbers (a,b), locate the point associated with the number a on the x-axis and through this point draw a line parallel to the y-axis. Then, locate the point associated with the number b on the y-axis and draw a line through this point parallel to the x-axis. There can be only one point of intersection for the two lines thus drawn. We associate this point with the ordered pair (a,b). Thus to an ordered pair of real numbers (a,b) there corresponds a unique point in the plane.

From the discussion above we can see that when graphing an ordered pair of numbers, the first element in the ordered pair (the abscissa) represents the number of units the point is to the right or left of the vertical axis. The second element in the ordered pair (the ordinate) indicates the number of units the point is above or below the horizontal axis.

To summarize, we say the Cartesian coordinate system sets up a one-to-one correspondence between the set of all ordered pairs of real numbers and all the points in the plane, and it provides us with a geometric model of the set of all ordered pairs of real numbers R X R.

When A and B are sets of real numbers, it can be helpful to picture relations such as A X B geometrically as a set of points in the Cartesian plane. It is possible to draw such graphs because A x B will always be a subset of R x R and the Cartesian plane is a model of all the ordered pairs of real numbers R x R.

The graphs of three Cartesian Products are shown in figure 4-3-4. The elements of each Cartesian Product are associated with unique points in the Cartesian plane.



A X B, where

A = {0, -1, -2} and
B = {-1, -2}

A X A, where

A = {-1, 0, 1}

A X B, where

A = {0, 1, 2} and

B = {0, 1, 2, 3}

Figure 4-3-4

Problem Set 4-3-5

1. Write a computer program which will generate the cross product of two finite sets of real numbers.

2. Given the following sets:

A = {-4, -2, 0, 2, 4}

B = {1, 3, 5}

C = {-2, -1}

(a) Use the computer program from problem 1 to generate the ordered pairs in the following Cartesian Cross Products:

(1) A X B

(2) B X A

(3) B X B

(4) A X C

(5) C X B

(b) Plot graphs of each of the relations in part a.

3. Sketch each of the following sets of ordered pairs on a Cartesian Coordinate System.

   a. {(0,3), (-2,5)}

   b. {(2,1), (2,3), (-4,1), (-2,5)}

   c. {(-2,3), (0,3), (-4,3),(4,3)}

   d. {(-2,1), (-4,3), (0,5), (2,5), (4,4)}

   e. {(5, -2), (3,-4), (1,0), (3,2), (5,4)}

4. Which of the sets of ordered pairs in problem 3 are subsets of A X B as generated in problem 2a?

5. Each of the following sets of ordered pairs represents the coordinates of the vertices of a geometric figure:

   I. {(5,7) (-5,7)(-5,11)(5,7)}

   II. {(-2,2)(6,2)(6,-2)(-2,-2)(-2,2)}

   III. {(0,3)(0,0)(-3,0)(-7,3)(0,3)}

   a. Plot each set of points on a graph and connect the points, in the order given, with straight lines.

   b. Determine the area of each figure formed.

Relations defined by equations and inequations are an important part of mathematics. Therefore, we give special attention to graphing them. Consider the relation $\{(x,y) | y = 2x - 4\}$. By selecting some values for x, we generate ordered pairs such as (1, -2), (-1,-6), (2,0) and (0,-4). The graph of these ordered pairs in the Cartesian plane is shown in Figure 4-3-6.

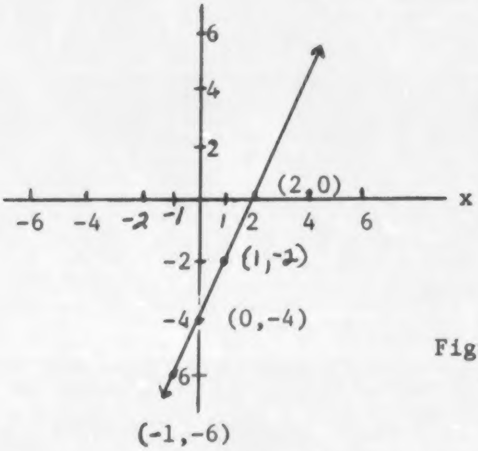


Figure 4-3-6

Note that these points appear to be in a straight line. It could be proven that all points of this relation lie in a straight line. Therefore we draw a line through these points as a means of graphing all the points of the relation. In Chapter 5 the topic of linear relations and their graphs will be investigated in more detail.

Problem Set 4-3-7

1. Sketch the graph of each of the following relations.

   a. $\{(x,y)|y = -3x - 2\}$

   b. $\{(x,y)|4x + y = -4\}$

   c. $\{(x,y)|y = -2x\}$

   d. $\{(x,y)|y = 8\}$

2. List 10 ordered pairs of the relation $\{(x,y)|y = |x|\}$. Include at least three pairs which have negative abscissas. Sketch the graph of this relation.

3. Write a BASIC program that will build a table of ordered pairs for the relation $\{(x,y)|y = \frac{1}{x^2 - 3x + 2}, -8< x <10 \text{ and } x\varepsilon I\}$.

   Run the program and use the information obtained to sketch the graph of this relation.

4. Write a BASIC program that will build a table of ordered pairs for the relation: $\{(x,y)|y = 2x^2 - 12x + 7\}$. Your table should contain $\{x|- 10 <x <10 \text{ and } x\varepsilon I\}$. Sketch the graph of the relation from the data you obtain.

5. Sketch the relation: $\{(-3,0), (6,-1), (-6,-1), (3,-2), (0,-1)\}$

To graph a relation in which the rule of formation is an inequation, we first graph the relation in which the rule of formation is the corresponding equation.

Example 4-3-8 Graph the relation:

$R = \{(x,y)|y\geq 3x - 1\}$

The rule of formation ($y \geq 3x - 1$) for the relation R is an inequation. Hence first we graph the relation $\{(x\cdot y)|y = 3x - 1\}$. (See figure4-3-9)

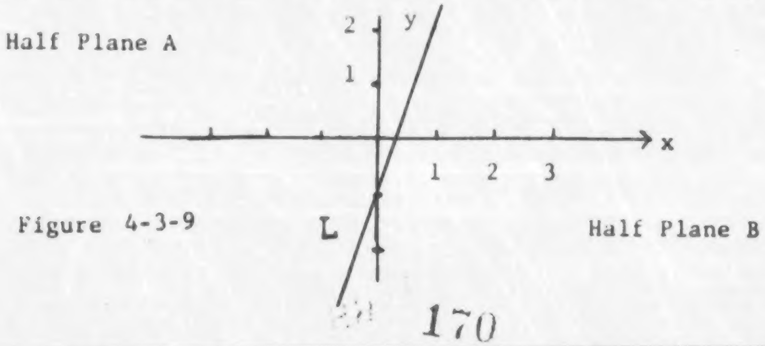


Figure 4-3-9

Notice that for each point on line L the y coordinate is EQUAL to three times the x coordinate less one. In general for any real number x the point with coordinates $(x, 3x - 1)$ is on line L.

The line L divides the Cartesian Plane into two half planes A and B. Let point P_1 with coordinates (x_1,y_1) be any point in half plane A. Draw line L_1 through point P_1 parallel to the y-axis. Let the intersection of line L with line L_1 be denoted by P_2. The coordinates of point P_2 will be (x_1,y_2) since all points on line L_1 will have x_1 for their first coordinate.

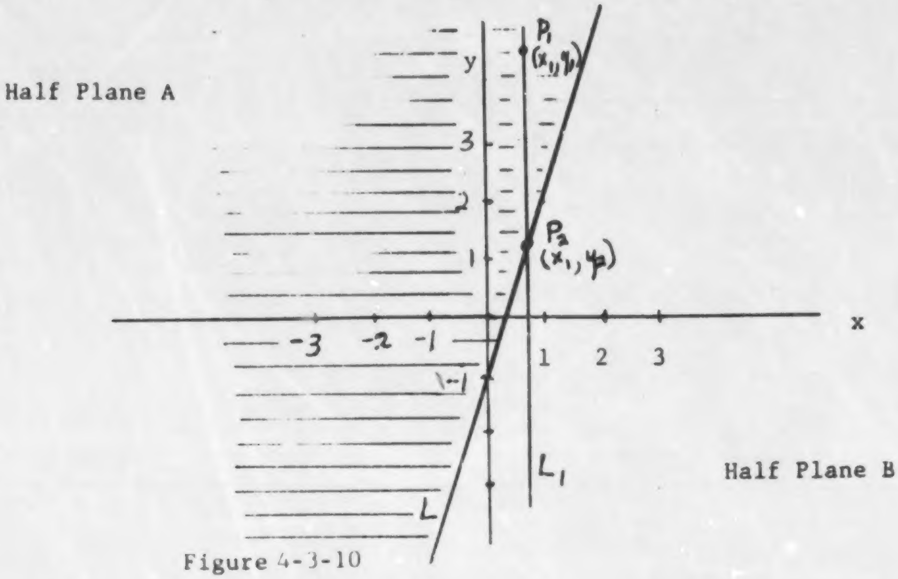


Figure 4-3-10

Since point P_1 is in half plane A as shown in Fig. 4-3-10 we can see that

$$y_1 > y_2$$

However

$y_2 = 3x_1 - 1$ because point P_2 is on line L.

Hence

$y_1 > 3x_1 - 1$ by substitution

This means that the ordered pair (x_1,y_1) is an element of the relation R,

$$R = \{(x,y)\,|\,y \geq 3x - 1\}$$

and that point P_1 is the graph of one of the ordered pairs in R.

Since P_1 is any point in half plane A we can conclude that every point in half plane A will have coordinates that belong to the relation $R = \{(x,y)\,|\,y \geq 3x - 1\}$

In a like manner we could show that any point in half plane B has a pair of coordinates that is NOT an element of $R = \{(x,y)\,|\,y \geq 3x - 1\}$.
Hence we conclude that the union of half plane A and line L is the graph of the relation $R = \{(x,y)\,|\,y \geq 3x - 1\}$.

Problem Set 4-3-11

1. Sketch the graph of the following relations.

    a. $\{(x,y)\,|\,y \le (1/3)x + 1\}$

    b. $\{(x,y)\,|\,y < -(1/3)x + 1\}$

    c. $\{(x,y)\,|\,2y > x - 3\}$

    d. $\{(x,y)\,|\,x < 4\}$

You may have found in working some of the problems from the last few sets that it was helpful to have the ordered pairs of real numbers arranged in some order prior to graphing. This is especially desirable if we must search through a large set of such pairs of numbers many times in order to find certain elements to be graphed. In some scientific problems, it is worthwhile to sort the same set of data several different ways and to have the sorted lists available for analysis.

We do a lot of sorting in our day-to-day living. Arranging the volumes of an encyclopedia on a shelf, the cards of a bridge hand, letters in a file (by date or by author), bills to be paid, and so on, all require sorting which we do manually. Most of the methods we use in these situations would be extremely slow if we needed to sort a large number of items instead of a dozen. For example, sorting the members of a senior class of 1000 students according to their grade average would be a slow and burdensome task. Let us consider several ways of sorting a large number of items in an efficient manner by computer. There are many algorithms for sorting. Some use very little storage space in the computer while sorting, but are rather slow. Others are very fast, but use more computer storage space. Still others are quite complicated but, by taking advantage of special properties of some particular computer, turn out to be very fast as well as economical in their storage requirements. We shall discuss one algorithm which is quite easy to understand and does not use much storage. It is not as fast in actual computation time as some other more complicated procedures. We shall also consider a procedure which allows one to take advantage of prior knowledge that the numbers to be ordered are somewhat in order to begin with.

The problem we shall consider may be stated as follows: Let A(1), A(2), A(3), . . ., A(N) be N numbers. Sort (i.e., rearrange) these numbers into increasing order. Before proceeding to the flow diagrams and the programs, however, we should discuss the strategy of the algorithm. One way to arrange numbers A(1),. . .,A(N) in increasing order is to set aside another area in storage as large as the area in which the numbers are presently stored. Then we could start by moving A(1) into the new area. If A(2) is greater than A(1), we could put it after A(1) in the new area; otherwise we could move A(1) over one place and put A(2) in front of it. Now we could determine where A(3) belongs and insert it in the proper place, moving the others over, if necessary. Eventually, we would have placed all the numbers into the new area. This method is used very often by card players who are dealt a hand for bridge or hearts, for example, and arrange their cards in order in their hand. This method is quite efficient for human beings because their eyes can very rapidly scan the cards which they already hold and because they can easily shift some of the cards to make room for the newcomer. In a computer, however, the frequent shifting of many numbers would take a much longer time than other methods would require, and the use of a second storage area as large as the original would be wasteful.

Our procedure would be improved if we would search for the smallest number among A(1), . . . , A(N) and move it to the new storage area, then search for the next smallest number, and so on. This would eliminate the shifting of numbers but would still require the second storage area. One small point should be noted before we leave the second method. After a number has been selected as being less than the remaining numbers, how do we remove it from further "competition"? We could delete it by closing up the remaining numbers, but this would introduce the shifting of numbers that we are trying to avoid. A more commonly used method is to replace the selected number by a very large number, usually the largest integer allowed in the language (in our case, 999999999). This guarantees that it will not be chosen again as the smallest number.

Everyone would probably agree that the second method is better than the first method because the shifting has been eliminated. This raises an important question, however. What are the criteria by which one decided which method is better? The usual criterion--other things being equal--is to minimize the number of comparisons that have to be made among pairs of numbers. This is a measure of the amount of computation that is needed and, therefore, a rough measure of the speed of the method. The other things that are assumed equal, however, are the shifting around of numbers in storage (which is really prohibitive in the first method) and the amount of extra storage needed. In the second method we have to compare each number with each other number; so there are N^2 comparisons. A third method which we shall discuss needs $\frac{N(N-1)}{2}$ comparisons, and it requires only one additional storage location, rather than a second area as large as the first.

In the third method we start with the first number A(1) and compare it with each of the others in turn. As long as it is less than or equal to the other number, it deserves to remain in the first position. As soon as another number is smaller than A(1), however, it becomes a better candidate for the "smallest-number" position, and it should be placed in position A(1) immediately. Since the original value of A(1) must be stored somewhere else now, the easiest way to handle it is to interchange the two numbers. If we now continue to compare the new A(1) with the rest of the other numbers, interchanging whenever necessary, we find that we end up with the smallest value in the entire set as the value of A(1). We may completely ignore the value of A(1) from now on, since it is in its final position already. What we need to do now is find the smallest of the remaining numbers, starting with A(2) and comparing with A(3), A(4), and so on. Each time we finish a string of comparisons, the number of comparisons to be made the next time decreases by 1. The first time when each of the other (N-1) numbers was compared with A(1), there were

N-1 comparisons. The second time there would be N-2 comparisons, and so on. The total of comparisons is (N - 1) + (N - 2) + (N - 3) + . . . + 1, which can be shown to have the value of $\frac{N(N-1)}{2}$. This will be shown in the chapter on sequences and series.

How do you interchange two numbers? If A(I) and A(J) are to be interchanged, we might try the statements

```
LET A(I) = A(J)

LET A(J) = A(I)
```

but, unfortunately, after the first statement is executed the value of A(I) will have been destroyed, and the second statement will not accomplish its purpose. It is necessary to preserve the value in A(I) before moving A(J) to A(I). If X is some available variable (storage location) we can write the interchange as follows:

```
LET X = A(I)

LET A(I) = A(J)

LET A(J) = X
```

This additional variable X is the only extra storage needed in the method since the same variable X may be used for all the interchanges.

We see in this algorithm loop which starts with the designation of A(1) as the first value against which comparisons are made. After A(1) is dealt with properly, we move to A(2) as the next value, for comparisons and so on. For each new value we have another loop, which consists of moving through the remaining values, interchanging when necessary. The set of numbers to be compared with the current value of A(I) always starts with A(I + 1) and goes through A(N). When A(N) becomes the value, there is no need for a comparison since it will be the largest value in the list.

The flow diagram for this algorithm is given in figure 4-3-12.

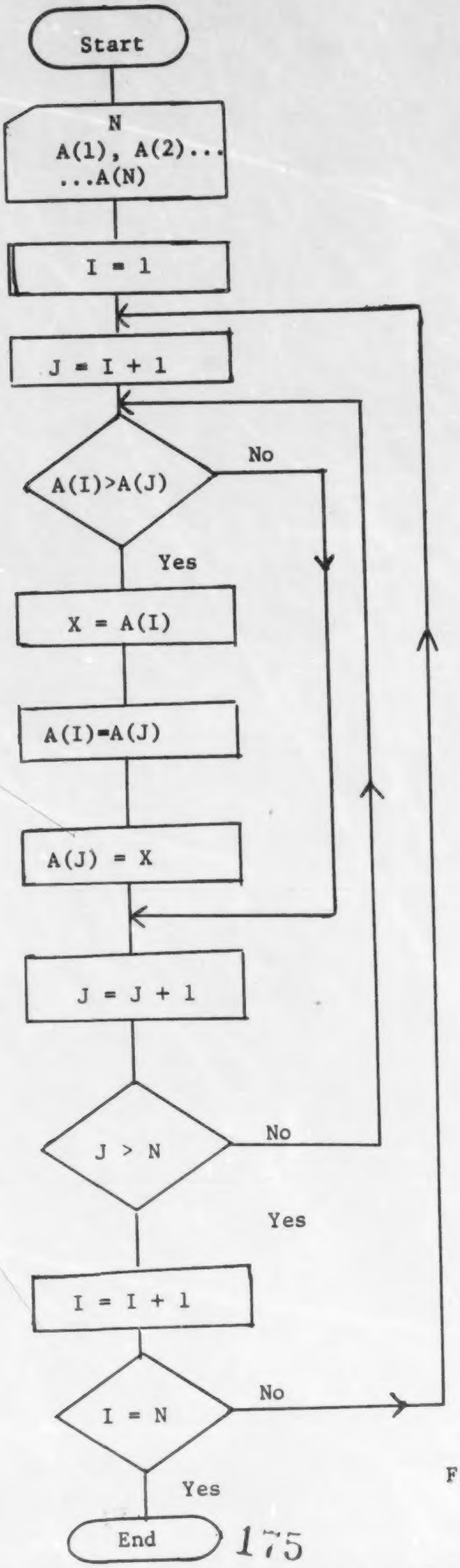

Start
N
A(1), A(2)...
...A(N)
I = 1
J = I + 1
A(I)>A(J)
No
Yes
X = A(I)
A(I)=A(J)
A(J) = X
J = J + 1
J > N
No
Yes
I = I + 1
I = N
No
Yes
End

Figure 4-3-12

Problem Set 4-3-13

Using the algorithm given above:

1. Write a BASIC program that will sort a list of numbers and arrange them from smallest to largest.

2. Use your program to sort the following list of numbers: 9, -3, 5, 1.

3. Trace your program using any other set of 4 numbers that you choose.

These sorting algorithms will be very helpful to us in graphing ordered pairs. We can arrange the pairs so the ordered pair with the smallest abscissa (first element) is first and the rest of the ordered pairs are arranged numerically by abscissas. If we then graph them in order from the smallest abscissa to the largest, we will proceed from left to right on our graph. Use this technique to work the following exercise.

Exercise 4-3-14

Sketch the graph of the following relation.
{(3,3), (-1,1) (0,2), (1,1), (-3,3), (3,2), (1,2), (-3,2), (3,1,), (1,3),}

These sorting algorithms will be most useful to us in future work, especially in the later part of this chapter.

4-4 Functions.

A common use of the computer in science, business, industry, etc., is to have it perform various operations and print results in tabular form. Suppose, for example, we are required to calculate the expected monthly cost of operating our school buildings and that we have decided to use a computer to assist us in the calculations. Since the monthly cost would depend, in part, on the number of school days in each month, it is necessary to provide the computer with information similar to that given in table4-4-1.

Monthly Count of School Days

Month	No. of School Days
January	20
February	19
March	23
April	17
May	20
June	9
July	0
August	0
September	19
October	21
November	19
December	13

Table 4-4-1

During the calculation of the expected cost, the computer would search the table for January and would then use the corresponding twenty days in the cost computation. Upon completing the computation for January, the machine would continue to February and compute the cost based on nineteen school days. To complete the calculation, this procedure would be repeated for each of the remaining months. We note the tabulated data which we used consisted of two sets - the set of months and the set of days. As we chose a particular month, from the first set, we find the corresponding number of days from the second set. To provide insight into the mathematical nature of this relation, it is helpful for us to model these sets as shown in Figure 4-4-1. M_i is the month and d_i the number of school days in that month.

$M_1 \quad d_1$

$M_2 \quad d_2$

$M_3 \quad d_3$

.

.

.

$M_{12} \quad d_{12}$

Figure 4-4-2

The important point here is to note that the two sets are related. They are in fact, related in the sense that to each M of the first set there corresponds exactly one d in the second set. A set of ordered pairs with such a correspondence is a relation which is called a function.

Although the inclusion of the month name in Table 4-4-1, is convenient for us, the month must be coded when used in the computer. If January is coded 1, February coded 2 and so on, the table would appear as in Table 4-4-3

Monthy Count of School Days

Month	No. of school days
1	20
2	19
3	23
4	17
5	20
6	9
7	0
8	0
9	19
10	21
11	19
12	13

Table 4-4-3

The set of ordered pairs {(1,20)(2,19),(3,23),(4,17),(5,20),(6,9),(7,0)(8,0),(9,19),(10,21),(11,19),(12,13)}is an example of what mathematicians call a function. Just as with other relations the set of all first elements is called the DOMAIN of the function, and the set of all second elements is called the RANGE of the function.

It will now be helpful for us to discuss the concept of a function more generally and to give it a mathematical definition.

Definition 4-4-4 Function

A function is a relation in which no two different ordered pairs have the same first element.

We emphasize that a function is a set of ordered pairs with a special condition placed upon each first element, namely that each first element must be associated with a single unique second element. It should be easy to see that not every set of ordered pairs is a function.

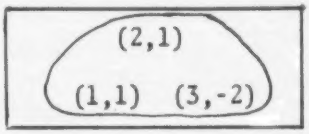


Figure 4-4-5

This relation is a function. Why?

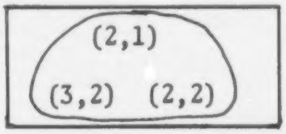


Figure 4-4-6

This relation is NOT a function. Why not?

According to the definition of a function each element in the domain is associated with one and only one element in the range. A function could be pictured in Figure 4-4-7

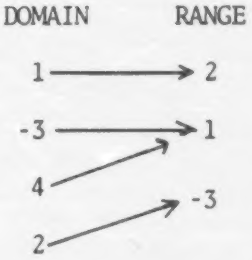


Figure 4-4-7

Figure 4-4-8 does not represent a function since for one element in the domain, namely 4, there is more than one corresponding element of the range.

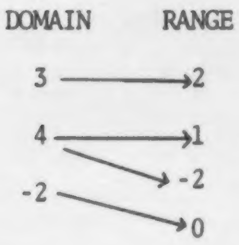


Figure 4-4-8

Consider now the correspondence between students and their grades on a particular test. This correspondence could be pictured as follows:

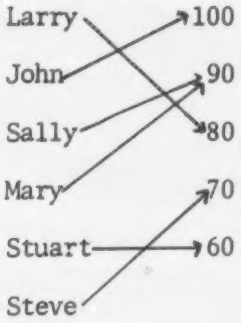


Figure 4-4-9

Exercise 4-4-10

Is the correspondence shown in Figure 4-4-9 a function? Why?/Why not?

Problem Set 4-4-11

1. Which of the following relations are functions? Justify your answer.

   a. $R_1 = \{(1,2), (3,4), (5,6), (7,8)\}$

   b. $R_2 = \{(1,2), (1,3), (1,4), (1,5)\}$

   c. $R_3 = \{(1,0), (0,-1), (-1,0), (2,-1)\}$

   d. $R_4 = \{(1,1), (2,2), (3,3), (4,4)\}$

   e. $R_5 = \{(0,0), (1,0), (0,1), (1,1)\}$

   f. $R_6 = \{(a,x), (b,y), (z,w), (b,x)\}$

   g. $R_7 = \{(x,y) | y = x + 1\}$

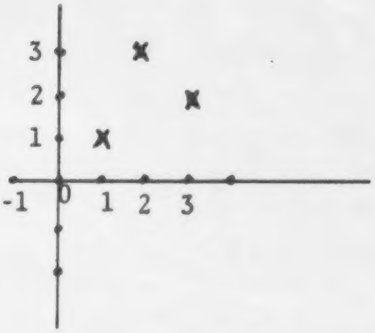
   h. $R_8 = \{(x,y) | y<x\}$

   i. $R_9 = \{(x,y) | y = |x|\}$

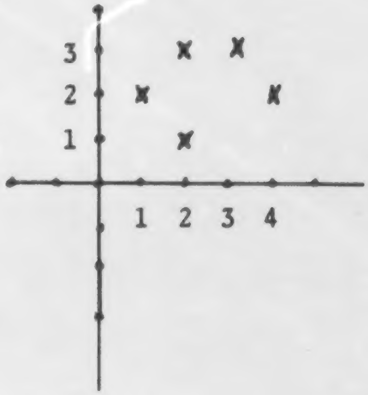
   j. $R_{10} = \{(x,y) | x = |y|\}$

2. Which of the relations whose graphs are given below are functions? Give the domain and range of each relation.

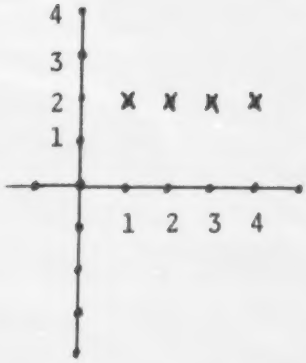
A



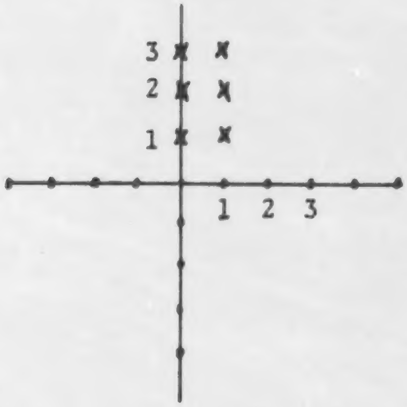
B



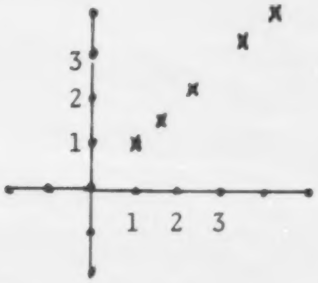
C



D



E



3. By a diagram show the correspondence between the first names and the last names of the authors of this book. Does this correspondence define a function. (Hint: In specifying a set, each element is listed only once.)

4. Sketch the graph of each of the following. Which of these are functions?

a. $\{(x,y)|y = 2x + 4\}$

b. $\{(x,y)|y = x\}$

c. $\{(x,y)|x = -3\}$

d. $\{(x,y)|y = \sqrt{x}\}$

e. $\{(x,y)|y^2 = x\}$

f. $\{(x,y)|y = 2\}$

g. $\{(x,y)|y \geq x\}$

h. $\{(x,y)|x^2 + y^2 = 36 \text{ and } x,y\varepsilon I\}$

Exercise 4-4-12

Write a BASIC program that will read in a set of ordered pairs and determine if the set is a function. Use the ordered pairs in Problem 4-4-11(1a-1e) above as data.

Many of the relations which are studied in mathematics consist of an infinite set of ordered pairs. For example, the relation $\{(x,y)\varepsilon R \times R|y = 1/2x^2\}$ is made up of an infinite number of ordered pairs (x,y) all of which satisfy tne equation $y = (1/2)x^2$. Considering this fact, we can see it would be difficult to determine whether or not the relation is a function by direct observation of its elements. Hence, we will develop a method for determining if a relation is a function by analysis of its graph.

The graph of the relation $\{(x,y)\varepsilon R \times R|y = (1/2)x^2\}$ is shown in Figure 4-4-13

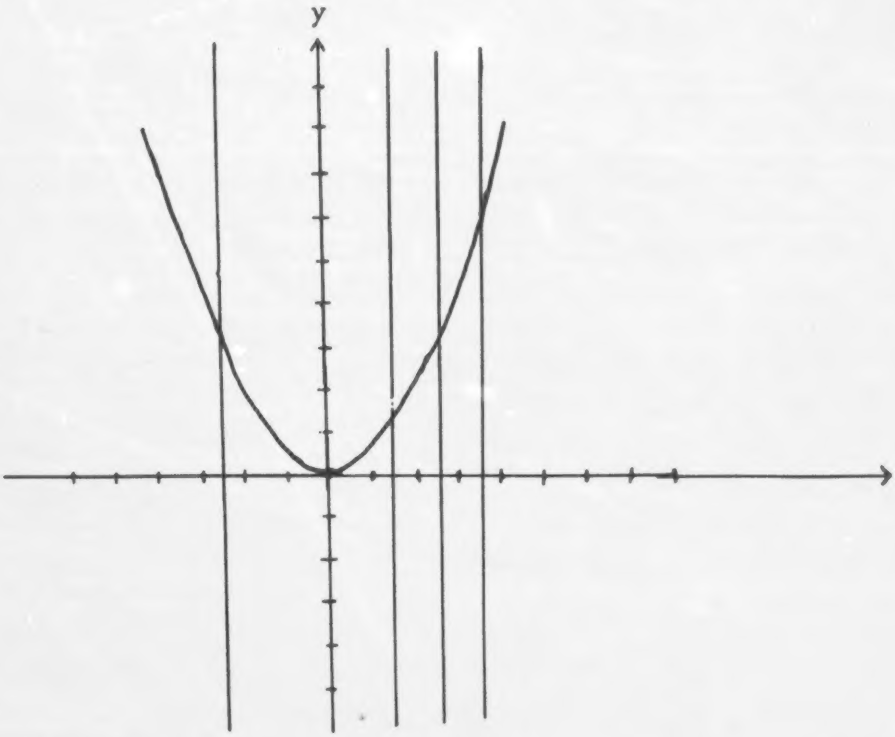


Figure 4-4-13

Now consider the set of all lines in the Cartesian plane which are parallel to the y-axis. A few of these lines are shown in Figure 4-4-13 If the intersection of each line in this set, with the graph of the relation , contains only one point, then the relation is a function. From Figure 4-4-13 we can see that each line on the Cartesian plane parallel to the y-axis will intersect the graph of the relation in only one point. Hence, the relation $\{(x,y)\varepsilon R \times R | y = x^2/2\}$ is a function.

Now consider the relation $\{(x,y)\varepsilon R \times R | x^2 + y^2 = 25\}$. The graph of this relation is shown in Figure 4-4-14

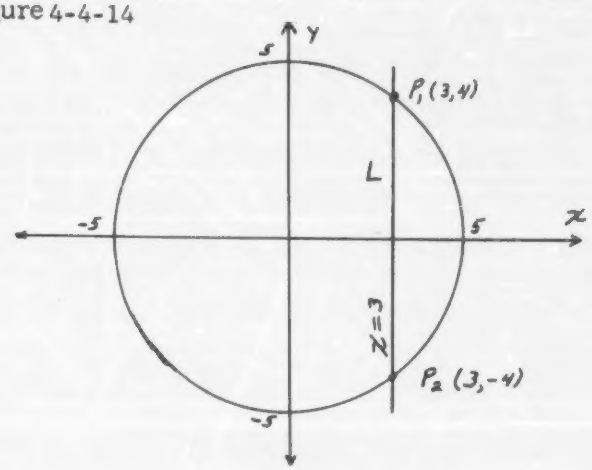


Figure 4-4-14

One element, L, of the set of all lines parallel to the y-axis is shown in Figure 4-4-14 Notice that the intersection of this line with the graph of the relation contains two points, P_1 and P_2. Hence, this relation is not a function because two different ordered pairs, (3,4) and (3,-4), have equal first elements.

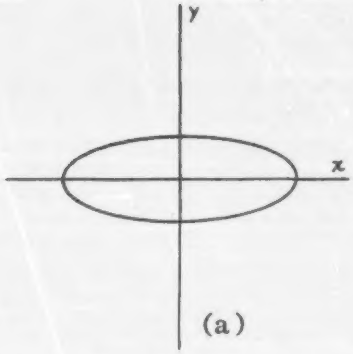
We can make the following generalization:

Given any relation and its graph:

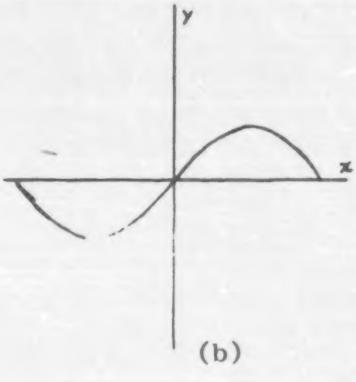
1. If the intersection of each line parallel to the y-axis with the graph of the relation contains no more than one point then the relation is a function.
2. If any intersection of any line parallel to the y-axis with the graph of the relation contains more than one point, then the relation is not a function.

Exercise 4-4-15

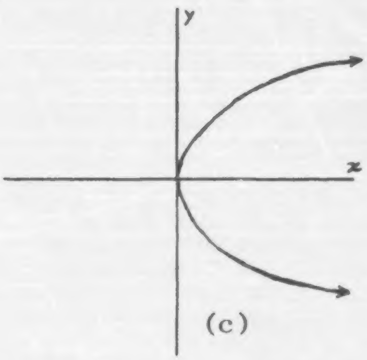
1. Graphs of some relations are shown below. Tell which are the graphs of functions:



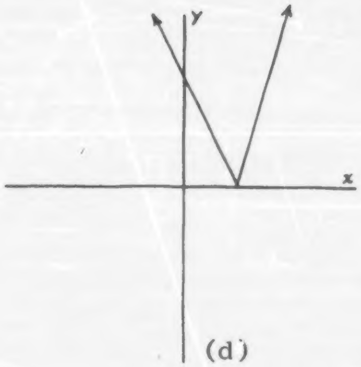
(a)



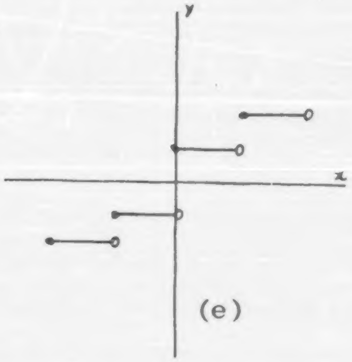
(b)



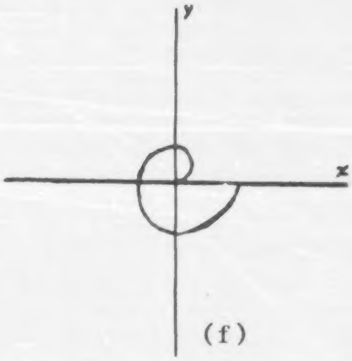
(c)



(d)



(e)



(f)

2. For $U = \{-2, -1, 0, 1, 2\}$, draw the graph of each of the following relations and tell which ones are functions.

   a. $\{(x,y) | y = x, x \varepsilon U, y \varepsilon U\}$

   b. $\{(x,y) | y = 2x + 1, x \varepsilon U, y \varepsilon U\}$

   c. $\{(x,y) | x = 2, x \varepsilon U, y \varepsilon U\}$

   d. $\{(x,y) | y < 2x + 1, x \varepsilon U, y \varepsilon U\}$

   e. $\{(x,y) | y = |x| + 1, x \varepsilon U, y \varepsilon U\}$

It is necessary to be able to define the domain of a function if it is described by the set builder notation rather than by actually listing the pairs of elements. Most of the functions we study in algebra will be subsets of R x R. Therefore, their domains will be subsets of the real numbers.

To find the domain of a function we determine that subset of the real numbers whose elements produce names for real numbers when substituted for the first component in the set selector.

Example: 4-4-16

The function $\{(x,y) \varepsilon RXR | y = 2x + 1\}$ has the set of real numbers for its domain. This is true because of the following fact:

> Any real number may be substituted for x in the expression $2x + 1$ and the result will be a name for some real number. For example, when seven is substituted for x, the result $2(7) + 1$ is a name for the number fifteen.

Example 4-4-17

The function $\{(x,y) \varepsilon RXR | y = 1/x\}$ has the following domain:

DOMAIN = $\{x | x \varepsilon R, x \neq 0\}$

The domain is the set of all real numbers excluding zero. This exclusion must be made because substitution of zero for x in the expression $1/x$ gives the result $(1/0)$ which is NOT a name for any real number. Hence, none of the ordered pairs of real numbers whose first element is zero can belong to the function and the domain does not contain zero.

Example 4-4-18

The function $\{(x,y) \varepsilon RXR | y = \sqrt{x}\}$ has the following domain.

DOMAIN = $\{x | x \varepsilon R, x \geq 0\}$.

The domain is the set of all positive real numbers and zero. The negative real numbers must be excluded because substitution of any of them for x in the expression $\sqrt{x}$ will result in an expression which is not a name for a real number. For example, substitution of -5 for x in $\sqrt{x}$ yields $\sqrt{-5}$. This is not the name of a real number.

From this point on in this text all relations, unless otherwise stated, are assumed to be sets of ordered pairs of real numbers. This point having been emphasized, we will now adopt the widespread practice of referring to relations and functions in RXR by their set selector equation. For example, the function $\{(x,y) \varepsilon RXR | y = 2x + 1\}$ may be referred to as "the function defined by $y = 2x + 1$."

The student will find many text books which refer to "the function $y = 2x + 1$." This can be misleading. Let us stress the point again that "$y = 2x + 1$" is not a function, it is an equation. The function is the set of all ordered pairs of real numbers which yield a true statement when substituted into this equation.

We now need to discuss some symbolism that has become standard notation. Lower case letters such as f, g, h, t, u, and others, are used to name functions. The function illustrated previously could be described as

$$f = \{(x,y) | y = 2x + 1\}.$$

A second symbol which has become standard through usage is one such as $f(x)$, $U(x)$ or other similar notation. $f(x)$, is read "f of x" or "f at x." It means the value of the function f for a given x.

In our example $f = \{(x,y) | y = 2x + 1\}$ to each element in the domain there is associated a unique element $f(x) = 2x + 1$.

If $x = 5$

then $f(5) = 2(5) + 1$

$f(5) = 11$

Therefore, 11 is the value of the function at $x = 5$.

The ordered pair representation (x,y) in f could be replaced by $(x,f(x))$ or by $(x,2x + 1)$ and the function could be described as $f = \{(x,f(x)) | f(x) = 2x + 1\}$

Example 4-4-19

1. $h = \{(1,2)(-3,5)(4,9)(2,3)\}$

   $h(1) = 2$

   $h(-3) = 5$

   $h(4) = 9$

   $h(2) = 3$

2. Given $g(u) = 2u + 3$ find $g(3)$, $g(0)$, $g(a)$, $g(a + 1)$

   $g(3) = 2(3) + 3 = 9$

   $g(0) = 2(0) + 3 = 3$

   $g(a) = 2(a) + 3 = 2a + 3$

   $g(a + 1) = 2(a + 1) + 3 = 2a + 2 + 3 = 2a + 5$

Problem Set 4-4-20

1. If $f(x) = 2x^2 + x - 1$, find:

   a. $f(2)$ b. $f(-1/2)$ c. $f(0)$ d. $f(a)$ e. $f(a + 2)$

2. If $g(h) = \frac{2h + 1}{2h - 1}$ find:

   a. $g(1)$ b. $g(\frac{a}{b})$ c. $g(d - 1)$

3. If $f(a) = a + 3$ and $g(a) = a - 3$ find:

   a. $f(0)$ c. $f(1) - g(1)$ e. $\frac{g(4)}{f(4)}$

   b. $3[g(0)]$ d. $f(3) \cdot g(3)$

Exercise 4-4-21

Write a BASIC Program to evaluate the following functions described

a. If $f(x) = 3x^2 + x + 2$ find $f(0)$, $f(3)$, $f(189)$, $f(-2)$, $f(-2067)$

b. If $g(x) = x^2 - 2x + 1$ find $g(0)$, $g(3)$, $g(189)$, $g(-2)$, $g(-2067)$

c. If $h(x) = \frac{1}{x^2 - x - 6}$ find $h(0)$, $h(3)$, $h(189)$, $h(-2)$, $h(-2067)$

Problem Set 4-4-22

1. If $r(x) = 3x - 1$ and $s(x) = (x + 1)$ find:

   a. $s(r(-1))$ b. $r(s(0))$ c. $s(r(a))$

Consider the example problem below:

Find the domain and range of g.

$$g(a) = \frac{1}{\sqrt{a - 4}}$$

Since the domain of this function is the set of all real numbers which yield names for real numbers when substituted for the variable a, $a - 4 > 0$, or $a > 4$. The domain D is thus: $D = \{a | a > 4\}$. The range is the set of all g(a). Since both the numerator and denominator of the expression $\frac{1}{\sqrt{a - 4}}$ are positive, g(a) must be positive and therefore the range is a set of positive real numbers.

2. Give the range for each of the functions defined below whose domain is the set of real numbers. If the domain is not the set of real numbers, give both the domain and the range of the function.

a. $f(x) = x$

b. $q(r) = \frac{r}{r - 1}$

c. $s(b) = b^2$

d. $q(d) = \frac{3d}{d^2 - 4}$

e. $t(y) = \frac{y^2 - 1}{y + 1}$

f. $f(x) = \sqrt{(x - 3)}$

g. $h(c) = -\sqrt{c}$

Given a relation r = {(1,2), (2,4), (3,6), (4,8)} there exists another relation s = {(2,1), (4,2), (6,3),(8,4)} obtained by interchanging each first and second component. For r, the domain is {1,2,3,4} and the range is {2,4,6,8}. For s, the domain is the range of r, and the range is the domain of r. From every relation r, we can get a corresponding relation s, called its converse relation, by interchanging the positions of the elements in each ordered pair in relation r.

Definition 4-4-23 Converse

The relation s is the converse of the relation r if and only if for every ordered pair (x,y) in r, there exists the ordered pair (y,x) in s.

Example: Let r = {(-1,0), (-1.2), (3,0), (5,2), (1,-1)}
then the converse s = {(0,-1), (2,-1), (0,3), (2,5), (-1,1)}

In the set builder notation, $\{(a,b) \mid (\text{set selector})\}$, the ordered pair (a,b) is an index which tells us which variable is the first coordinate and which is the second coordinate. If we interchange the elements of the index and keep the set selector the same, we have $\{(b,a) \mid (\text{set selector}\}$. The result is to interchange each first and second coordinate in the set. This is one way to produce the converse of a relation. For example, if $r = \{(x,y) \mid y = 2x\}$, then the converse s is $\{(y,x) \mid y = 2x\}$. Some of the ordered pairs in r are $(1,2)$, $(-1,-2)$, $(2,4)$, $(1/2, 1)$ etc. and because the elements of the index, (x,y) have been interchanged s would contain the ordered pairs $(2,1)$, $(-2, -1)$, $(4,2)$, $(1, 1/2)$ etc.

Consider the relation $t = \{(a,b) \mid b = 2a\}$. Its converse s could be written as $s = \{(b,a) \mid b = 2a\}$. Compare now $s = \{(b,a) \mid b = 2a\}$ and $s' = \{(a,b) \mid a = 2b\}$.

Exercise 4-4-24

1. What are some ordered pairs which belong to s?

2. What are some ordered pairs which belong to s'?

3. Does each ordered pair which belongs to s also belong to s'?

4. Are s and s' the same set?

Since s is the converse of t and s = s' it follows that s' is the converse of t. Now compare t and s'.

$$t = \{(a,b) \mid b = 2a\}$$
$$s' = \{(a,b) \mid a = 2b\}$$

What do you notice about the indices? Set selector?

We conclude that one of the ways to describe the converse of a relation is to interchange the variables in its set selector.

Example: If $r = \{(m,n) \mid n = 2m - 2\}$ then the converse of r is $s =\{(m,n) \mid m = 2n - 2\}$. Usually we like to express the set selector of a relation in such a way that it tells how the second element of an ordered pair is determined from a given first element. If we take the set selector $m = 2n - 2$ and solve for n, we get $n = (1/2)m + 1$. Then s would be $\{(m,n) \mid n = (1/2)m + 1$.

Exercise 4-4-25

1. Write the converse of each of the following relations in set builder notation. Solve the set-selector for the second element as in the example above.

   (a) $\{(x,y) \mid y = 2x - 3\}$

   (b) $\{(x,y) \mid 3x - 2y = 4\}$

When the domain of a function has conditions placed on it, care must be taken to determine the proper conditions to be placed on the domain of the converse. For example, the relation $g = \{(a,b) | b = 2a + 4, 1 \le a \le 4, a \epsilon I\}$ consists of the ordered pairs (1,6), (2,8), (3,10) and (4,12). The converse relation would contain the ordered pairs (6,1), (8,2), (10,3) and (12,4). Note that the domain of the converse is {6,8 10,12}, therefore the converse relation would be $\{(a,b) | b = (1/2)a - 2, 6 \le a \le 12, b \epsilon I\}$.

Exercise 4-4-26

1. Write the converse of each of the following relations. State the domain and range of each converse.

a. $\{(x,y) | y = 2x - 1\}$

b. $\{(x,y) | x - 2y = 3\}$

c. $\{(x,y) | y = 1/x\}$

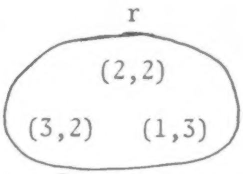
d. $\{(x,y) | xy = 2\}$

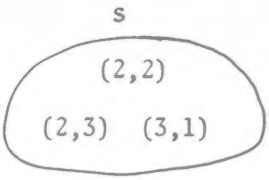
e. $\{(x,y) | y = x^2, x \ge 0\}$

f. $\{(x,y) | y^2 = -x^2 + 9\}$

Since a function is a special kind of relation, every function has a converse. We can see that the converse of a function may or may not be a function.

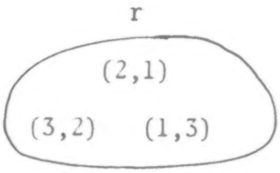
As an example:

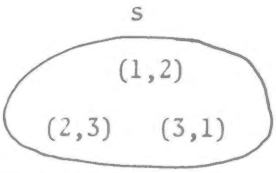




r is a function, its converse, s, is not.

But -





r is a function and its converse, s, is also a function.

Definition 4-4-27 Inverse

The converse s of function f is an inverse denoted f^{-1} if and only if the converse s is a function.

In the symbol f^{-1}, the -1 is not an exponent. It is merely a symbol denoting the inverse of the function f. We could have used the symbol f^{*} or $f^{\$}$ or something similar, but f^{-1} is the notation commonly used.

In the event the converse of a function is not a function, we do not assign this converse any special notation or name, it is just another relation.

Example 4-4-28

1. $f = \{(x,y) \mid y = x^2,\ -2 \le x \le 2, x \in I\} = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$

   then its converse is:

   $s = \{(x,y) \mid x = y^2,\ -2 \le y \le 2, y \in I\} = \{(4,-2),(1,-1),(0,0),(1,1),(4,2)\}$

   but

2. $f = \{(x,y) \mid y = 2x + 1,\ -2 \le x \le 2,\ x \in I\} = \{(-2,-3),(-1,-1),(0,1),(1,3),(2,5)\}$

   $f^{-1} = \{(x,y) \mid y = (1/2)(x - 1),\ -3 \le x \le 5, y \in I\} = \{(-3,-2),(-1,-1),(1,0),(3,1),(5,2)\}$

In example one, we can see the converse s is not a function. In example 2, f^{-1} is the inverse of f.

Exercise 4-4-29

Write a BASIC Program that will print out:

1. The converse of a finite relation.
2. Apply this program to the following relations:
   a. $\{(1,1),\ (2,2),\ (3,3),\ (3,5)\}$
   b. $\{(1,1),\ (2,4),\ (3,9),\ (-1,1),\ (-2,4),\ (-3,9)\}$
   c. $\{(1,4),\ (2,1),\ (3,2),\ (4,5)\}$

Problem Set 4-4-30

1. Which of the following relations are functions and which functions have inverses? If there is an inverse, give its domain and range.

   a. $\{(8,1), (7,2), (6,3), (5,4)\}$

   b. $\{(1,4), (5,6), (9,4), (3,7), (9,6)\}$

   c. $\{(3,2), (7,5), (4,9), (3,5)\}$

   d. $\{(3,2), (4,5), (6,4), (7,8), (10,9)\}$

2. Sketch the graph of f and f^{-1} and give the domain and range of each.

   a. $f(x) = 2x + 6$

   b. $f(s) = 5s + 3$

   c. $f(x) = 3 - (1/2)x$

   d. $f(t) = (t/2) + 1$

3. For each of the following functions, graph f and its converse. Is the converse an inverse of f?

   a. $f = \{(x,f(x)|f(x) = 2x - 1\}$

   b. $f = \{(x,f(x)|f(x) = |x + 2|\}$

   c. $f = \{(x,f(x)|f(x) = x^2 - 1 \text{ and } x \geq 0\}$

   d. $f = \{(x,f(x)|f(x) = \sqrt{x}\}$

   e. $f = \{(x,f(x)|f(x) = \sqrt{x^2 - 4}\}$

   f. $f = \{(x,f(x)|\ f(x) = -\sqrt{4 - x^2}\ \}$

4. For problems 3a,b,d,e, and f above, restrict the domain to be $\{x|-4<x<4 \text{ and } x\epsilon I\}$. Write a program that will cause the computer to print f^{-1}, if it exists.

4-5 Functions in BASIC

In solving mathematical problems, it is often necessary to take the square root or absolute value of a number, or to find the integer part of a given number. Because procedures such as these are needed quite often, the BASIC language includes a set of prewritten programs called LIBRARY FUNCTIONS which are used to calculate the values for certain functions. Some of these functions are listed in the following Table 4-5-1

Table 4-5-1

FUNCTION	BASIC NAME	RESULT
Square Root	SQR (x)	Takes the square root of x if $x > 0$. If $x < 0$, the computer takes the square root of its absolute value.
Absolute Value	ABS (x)	Takes the absolute value of x.
Integer Part	INT (x)	Gives the largest integer which is not greater than x. For example, if x = 9.8 INT (x) = 9, If x = -3.33 INT (x) = -4.
Exponential, base e	EXP (x)	Gives the value obtained by raising e to the x^{th} power.
Sine	SIN (x)	Gives the sine of x where x is expressed in radians.
Cosine	COS (x)	Gives the cosine of x where x is expressed in radians.
Tangent	TAN (x)	Gives the tangent of x, where x is expressed in radians.
Inverse tangent'	ATN (x)	Gives the inverse tangent of x, where x is a real number.
Logarithm, base e	LOG (x)	Gives the natural logarithm of X. If x is a negative, the computer takes the logarithm of its absolute value.

In this section, we will be concerned only with the SQR, ABS, and INT, functions. The other functions will be discussed later in the text.

In a BASIC program, we can direct the computer to use a library function by writing the three letter name of the function, followed by an expression enclosed in parentheses. The expression can be given explicitly as in SQR (3) or implicitly as in ABS (X + Y). The expression in parentheses is called the ARGUMENT of the function. For example, if we want the computer to print the square root of seven, we could write the following instructions:

```
10  LET S = SQR (7)

20  PRINT S

30  END
```

In this case, the argument for the square root function is seven. The argument can be any BASIC numeral, variable, or expression. To print out the square roots of positive integers less than eleven, we would write:

```
10 FOR N = 1 TO 10

20 LET S = SQR (N)

30 PRINT S;

40 NEXT N

50 END
```

We can also write expressions such as SQR(3*X + 5), SQR(A↑2 + B↑2) and SQR(3*Y↑4 - 2) in which the arguments are $3x + 5$, $A^2 + B^2$ and $3X^4 - 2$ respectively. When arguments like these are used, the computer evaluates the argument first and then calculates the square root of this number.

It should be noted that $\sqrt{X}$ and SQR(X) are not identical. If $x<0$, $\sqrt{x}$ is not a real number. In BASIC, if $x<0$ the argument of SQR is negative, and the machine computes the SQR of the absolute value of the argument. Fortunately, when this happens the computer types out the message:

```
SQUARE ROOT OF A NEGATIVE NUMBER IN N
```

where N is the line number of the statement containing the SQR function.

Exercise 4-5-2

Write a BASIC program using the SQR function to compute and print out a table of squares and square roots for the positive odd integers less than 26.

In BASIC, we may define other functions by means of the DEF function. This is especially useful when we use a given function several times in one program. The name of the defined function must be three letters, the first two of which must be F and N with the third letter being chosen arbitrarily. It should be noted, however, that this letter should not be used elsewhere in the same program as a variable. Thus, you may define up to 26 functions - e.g. FNA,FNB, etc. Suppose we are required to compute the hypotenuse of several right triangles given the length of the other sides, we could define a function called FNH as follows:

```
10 DEF FNH (A,B) = SQR(A↑2 + B↑2)
```

and later call for various values of the function this way:

```
 10 DEF FNH (A,B) = SQR(A↑2 + B↑2)
100 LET S = FNH(3,4)
110 PRINT S
120 LET J = FNH(S,A + 4)
130 PRINT J
```

The DEF statement may occur anywhere in the program, and the expression to the right of the equal sign may be any formula which can be fitted onto one line. It may include any combination of other functions, including ones defined by other DEF statements.

Exercise 4-5-3

Using the DEF function, write a BASIC program to compute the area of a triangle given the length of the three sides. (HINT: Remember HERO'S FORMULA!)

4-6 Continuous and Discontinuous Functions.

Many functions in the set of real numbers have graphs that are continuous curves, that is without jumps or holes in them. Such functions are called CONTINUOUS FUNCTIONS. Graphs of two functions are shown below. One is continuous and one is not.

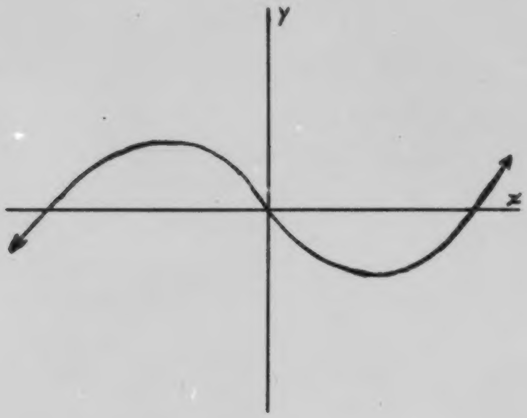


Figure 4-6-1

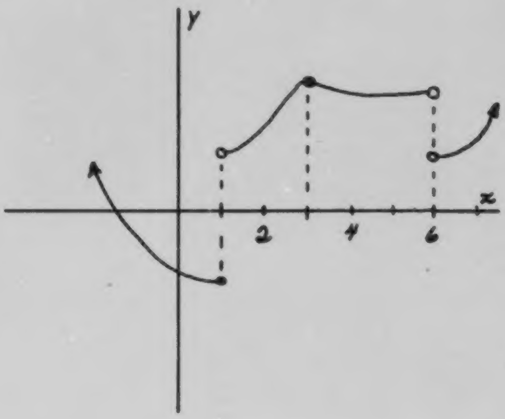


Figure 4-6-2

Figure 4-6-1 illustrates a function which is continuous, while Figure 4-6-2 shows a function that is not continuous. In the discontinuous function there is a jump at $x = 1$, a hole at $x = 3$ and both a jump and hole at $x = 6$. Any of these situations is sufficient to say that the function is discontinuous. We say there is a discontinuity at $x = 1$, $x = 3$ and $x = 6$. If we examine the function graphed in Figure 4-6-2, we see that the function is continuous on certain sets of real numbers called intervals. For example, if we restrict the domain of the function to be $\{x|1<x<3\}$ then the function is continuous on that interval but the function is discontinuous on the restricted domain $\{x|0<x<6\}$.

Exercise 4-6-3

Name two more intervals for which the function in Figure 4-6-2 is (a) continuous, (b) discontinuous.

Any set of real numbers such as $\{x|a\le x\le b\}$ or $\{x|a<x<b\}$ is called an INTERVAL; the former is called a CLOSED interval, since it contains the endpoints a and b. The latter is called an OPEN interval since it does not contain these endpoints. Some intervals might be closed on one end but open on the other. Such sets would be of the form $\{x|a\le x<b\}$ or $\{x|a<x\le b\}$.

Exercise 4-6-4

Refer again to Figure 4-6-2 Tell whether the function is continuous or discontinuous on each of the following intervals:

a. $\{x|1\leq x<3\}$

b. $\{x|1<x<3\}$

c. $\{x|3<x<6\}$

d. $\{x|3<x\leq 6\}$

In summary we can say that for a function to be continuous on an interval, f(x) must exist for every x on that interval. For example, the function $f = \{(x,f(x))|f(x) = \frac{1}{x - 2}\}$ is not continuous on the interval $\{x|0<x<3\}$ because f(2) is undefined in the set of real numbers. The function $g = \{(x,g(x)|g(x) = x^3 - x\}$ is continuous on all intervals because g(x) is a real number when any real number is substituted for x. (See Figures 4-6-5 and 4-6-6).

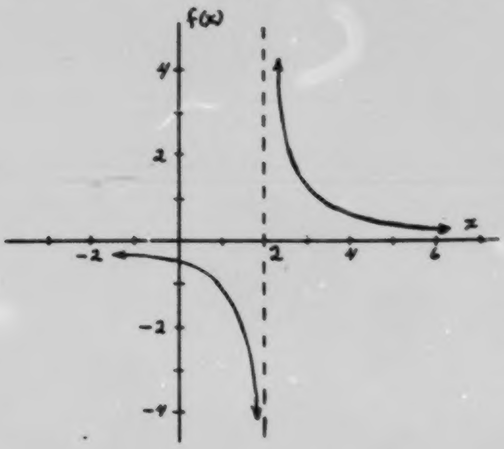


Figure 4-6-5

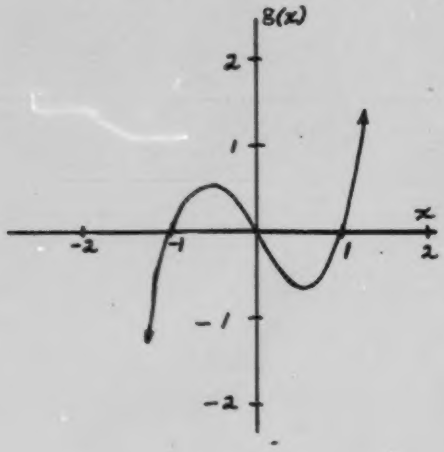


Figure 4-6-6

The dotted line in Figure 4-6-5 is called an ASYMPTOTE. The asymptote is used to indicate the number for which the function is undefined.

Exercise 4-6-7

1. Graph each of the following functions.
Give domain and range of each.
State whether each is continuous or discontinuous.
Name the points of discontinuity, if any.

$f = \{(x,y) \mid y = 2x^2 - x\}$

$g = \{(x,y) \mid y = \frac{x}{x - 3}\}$

$h = \{(x,y) \mid y = \frac{|x|}{x}\}$

$j = \{(x,y) \mid y = [|x|] - x, 0 < x \leq 3\}$

$r = \{(x,y) \mid y = \frac{2}{x^2 - 4}\}$

$s = \{(x,y) \mid y = x^4 - 8x^2 + 16\}$

$t = \{(x,y) \mid y = \frac{-x - 2}{x - 1}\}$

$u = \{(x,y) \mid y = \sqrt{4 - x^2}\}$

2. Name the smallest interval that would contain all the points of discontinuity of functions j, r, and u above.

3. Write a computer program that would test the function defined by $f = \frac{1}{25x^2 + 5x - 3}$ for points of discontinuity over the interval $\{x \mid -1 < x < 1\}$ in increments of .1. If no points of discontinuity are found in this test, have the computer print that the function is continuous, have it print out the points of discontinuity found.

We must point out that testing a function for continuity at a finite number of points as we did in Problem 3, is not a valid test for determining continuity on a real interval. On any interval of the real numbers there is an infinite set of reals therefore testing all points one at a time would be an impossible task, even for the computer. In a branch of mathematics known as analysis, techniques not applicable to the computer are developed for making valid tests of continuity of a function on an interval.

4. Adjust your program in Problem 3 above to test the following functions in increments of .1 over the interval specified.

$f = \{(x,y) \mid y = \frac{1}{x^2 - .2x}\}$ for $\{x \mid -\frac{1}{2} < x < \frac{1}{2}\}$

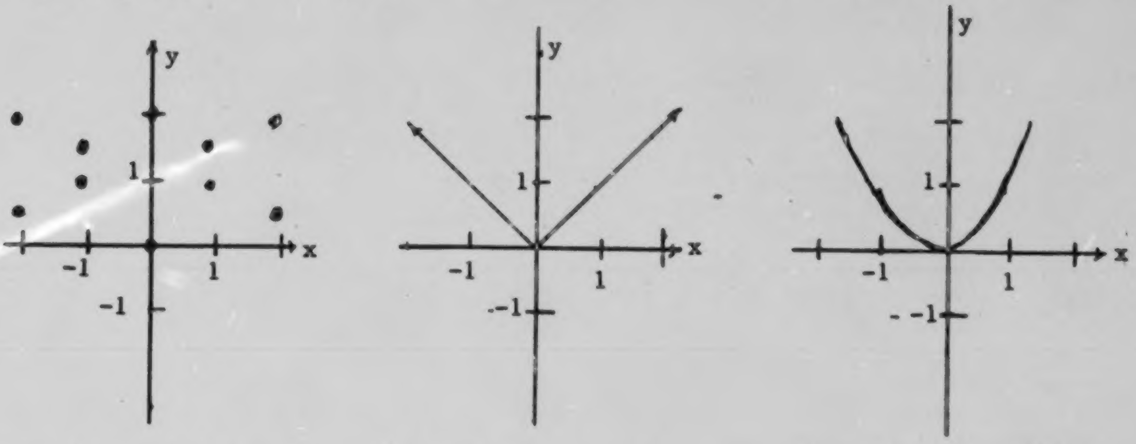
$g = \{(x,y) \mid y = \frac{x + 1}{x^2 + 2.3x + .6}\}$ for $\{x \mid 0 < x \leq 2.5\}$

$h = \{(x,y) \mid y = \frac{x}{15x^2 + x + 2}\}$ for $\{x \mid -.5 \leq x \leq \frac{2}{5}\}$

4-7 Symmetry

Symmetry is often found in nature and in many objects created by man. Most people have an intuitive understanding of symmetry but would have difficulty giving a meaningful description or definition of this property. In this section we will study symmetry from the mathematicians's viewpoint. We will be concerned primarily with the property of symmetry as applied to relations.

In order to develop a mathematical definition of symmetry we will begin with our intuitive feeling for this property and examine three symmetric relations. Consider the relations f, g and h whose graphs are shown in Figure 4-7-1 Each of these graphs is symmetric with respect to the y-axis.



f = $\{(0,0)(1,1)(-1,1)$
$(2,2)(-2,2)(-2, 1/2)$
$(2, 1/2)\ (-1, 1\frac{1}{2})(1,1\frac{1}{2})$

$(0,2)\}$

g= $\{(x,y)\,|\,y = |x|\}$

h = $\{(x,y)\,|\,y = x^2\}$

Figure 4-7-1

These relations are said to be symmetric with respect to the y-axis because the portion of the graph to the left of the y-axis is the "reflection" or "mirror image" of that portion of the graph to the right of the axis. If we were to place a small plane mirror along the y-axis in any of the graphs and view the reflecting surface from the side we would see the total graph of the relation. Part of it would be on the paper and part would appear as an image in the mirror. This is an experiment you can try at home

Observing the graphs of f, g and h gives us an intuitive understanding of symmetry with respect to the y-axis. However, we are more interested in finding some mathematical description of y-axis symmetry. Notice that for any point P_1 in the graph of relation f, there exists a point P_2 on the opposite side of the y-axis and equidistant from it. For example, the points $(1, 3/2)$ and $(-1, 3/2)$ in the relation f are on opposite sides of the y-axis and equidistant from it. In fact, for any point $(a,b)\in f$, there is a corresponding point $(-a,b)\in f$ so that (a,b) and $(-a,b)$ are on opposite sides of the y-axis and equidistant from the y-axis. This observation applies equally to all three relations f, g and h. Before reading further, see if you can write a definition of y-axis symmetry.

Definition 4-7-2 Y-axis Symmetry

A relation r is symmetric with respect to the y-axis if and only if for each $(a,b)\in r$, there exists $(-a,b)\in r$. The y-axis is called the axis of symmetry.

Example 4-7-3

$r = \{(1,3)(-1,3)(2,9)(4,7)(-2,9)(-4,7)\}$ r is symmetric with respect to the y-axis because

$(1,3)$ and $(-1,3)$ are both elements of r and

$(-2,9)$ and $(2,9)$ are both elements of r and

$(4,7)$ and $(-4,7)$ are both elements of r and

there are no other elements in r.

Example 4-7-4

$s = \{(-3,6)(3,6)(2,1)(-2,4)(6,3)(-6,3)\}$

s is not symmetric with respect to the y-axis because

$(2,1)\in s$ but $(-2,1)\notin s$.

Is there another reason why s does not have y-axis symmetry?

Example 4-7-5

$t = \{(-13,4)(7,2)(6,9)(-6,9)(13,4)(-6,2)\}$. t is not symmetric with respect to the y-axis because

$(7,2)\in t$ but $(-7,2)\notin t$.

Is there another reason why t does not have y-axis symmetry?

Example 4-7-6

Prove that $u = \{(x,y) | y = x^2\}$ is symmetric with respect to the y-axis.

In order to prove that the relation u has y-axis symmetry, we must show that for any $(a,b) \epsilon u$, there exists $(-a,b) \epsilon u$.

Proof:

Given $U = \{(x,y) | y = x^2\}$

For any $(a,b) \epsilon U$,

$b = a^2$ Because (a,b) satisfies the set selector $y = x^2$.

However $a^2 = (-a)^2$ $\forall_x \quad x^2 = (-x)^2$

Therefore $b = (-a)^2$ by substitution.

Hence $(-a,b) \epsilon U$ because (-a,b) also satisfies the set selector $y = x^2$.

$\therefore$ For any $(a,b) \epsilon U$ there exists $(-a,b) \epsilon U$.

U is symmetric with respect to the y-axis.

Exercise 4-7-7

1. Which of the following relations are symmetric with respect to the y-axis?

   a. $\{(x,y) | xy = 12\}$
   b. $\{(x,y) | y = |x| - 3\}$
   c. $\{(x,y) | x^2 + y^2 = 16\}$
   d. $\{(x,y) | y = -x^2\}$
   e. $\{(x,y) | 2y + 3x = 14\}$
   f. $\{(x,y) | y = 3x\}$
   g. $\{(x,y) | |x| + |y| = 6\}$

2. Prove that the following relations are symmetric with respect to the y-axis by the method outlined in Example 4-7-6.

   a. $\{(x,y)|y = (x + 2)(x - 2)\}$

   b. $\{(x,y)|y = |x|\}$

   c. $\{(x,y)|x^2 - y^2 = 3\}$

The subject of symmetry is important because it gives us a powerful tool for graphing relations. If we can determine that a relation has y-axis symmetry by analysis of the set selector, half the graph can be sketched quickly by viewing the other half. This idea will be expanded later, along with further development of the topic of symmetry. In the meantime we will attempt to build a computer program to test a relation for y-axis symmetry.

Let us begin by making a few preliminary decisions concerning the program we want to construct. First we will agree to design our computer program to check a finite relation, R. Assume that this finite relation, R, contains "N" ordered pairs that have been stored in the computer under the subscripted variables X(I) and Y(I).

i.e. $R = \{(X(1),Y(1)),(X(2),Y(2)), \cdots ,(X(N),Y(N))\}$

In order to check the relation R for y-axis symmetry, every ordered pair in R must be compared with every other ordered pair in R to determine that for each $(X(I), Y(I)) \varepsilon R$ there exists $(-X(I), Y(I)) \varepsilon R$. This requirement of comparing every element in a set with every other element in the same set brings back the remembrance of double nested logic loops in Chapter I. (Remember the section on rational numbers?) Figure 4-7-8 shows this double nested logic. Study the flow chart carefully.

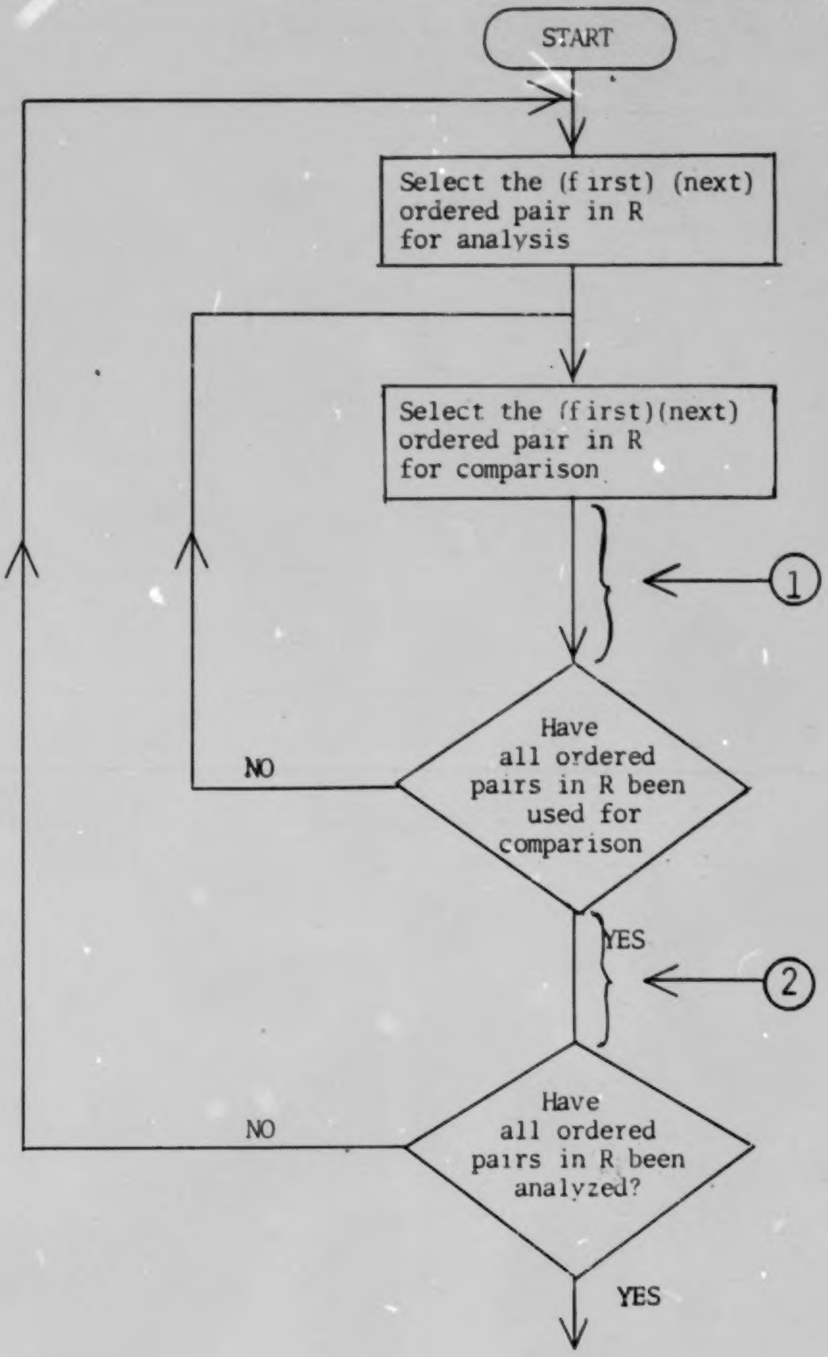

START
Select the (first) (next) ordered pair in R for analysis
Select the (first)(next) ordered pair in R for comparison
①
Have all ordered pairs in R been used for comparison
NO
YES
②
Have all ordered pairs in R been analyzed?
NO
YES

Figure 4-7-8

We can now analyze the logic required in section ① of the flow chart. Once two ordered pairs from R have been selected for comparison, what decisions must be made concerning the coordinates of these ordered pairs? First, we must find out if the ordered pairs have equal second coordinates. Secondly, when the second coordinates are equal we must know if the first coordinates are additive inverses of each other. This logic is shown in Figure 4-7-9

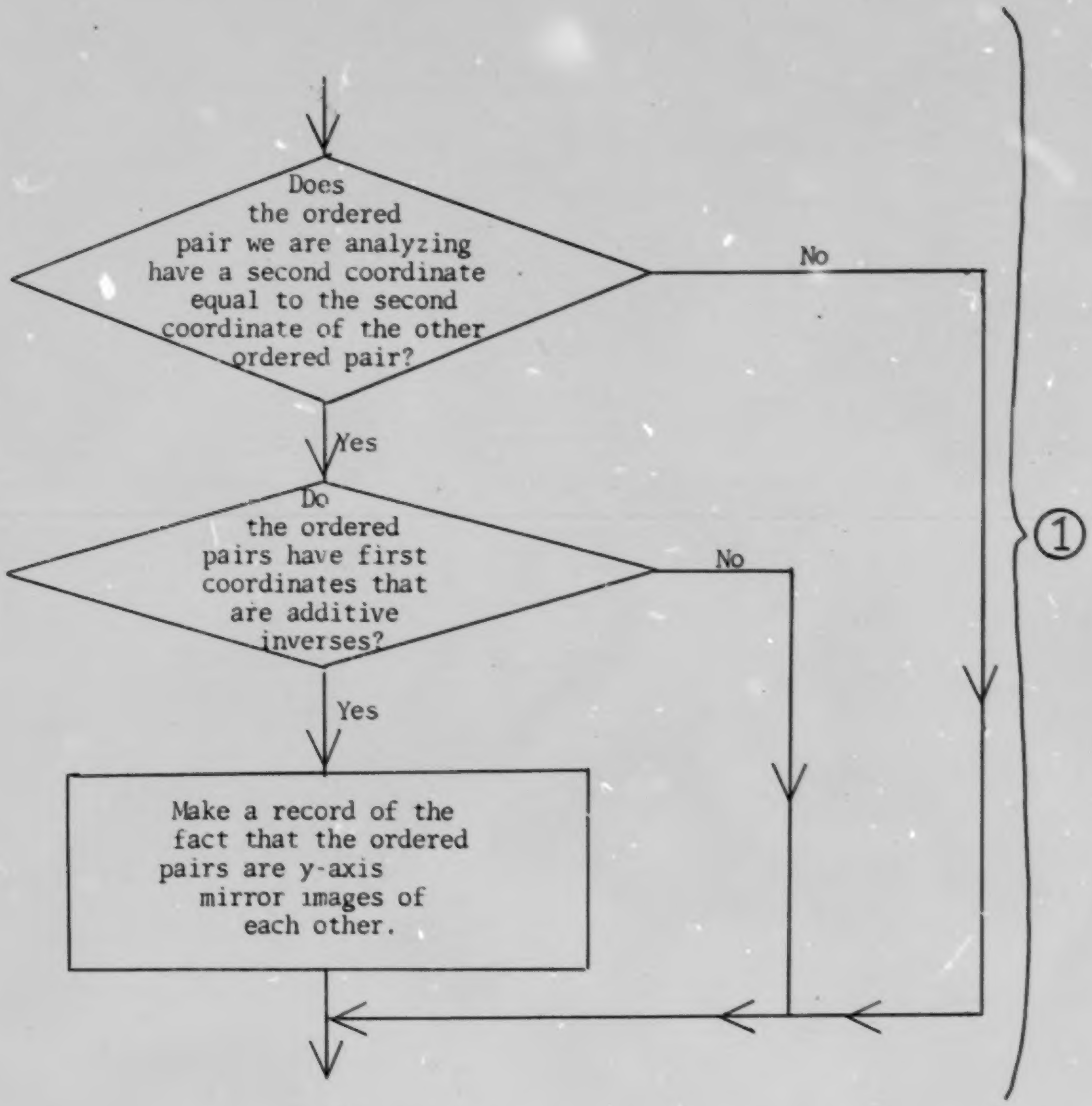

Does the ordered pair we are analyzing have a second coordinate equal to the second coordinate of the other ordered pair?
No
Yes
Do the ordered pairs have first coordinates that are additive inverses?
No
Yes
Make a record of the fact that the ordered pairs are y-axis mirror images of each other.
①

Figure 4-7-9

The analysis of the logic required in Section ② of the original flow chart in Figure4-7-8 is very simple. If the ordered pair we have analyzed does not have a mirror image we print out a message that the relation R does not have y-axis symmetry and stop.

Exercise 4-7-10

1. Draw a flow chart for the logic of Section ② as described in the paragraph above.
2. Write a computer program which will READ a finite relation from a data statement into the variables X(I) and Y(I). Your program is to determine if the relation has y-axis symmetry.
3. Test your program on the following relations:
    - a. {(1,3)(-1,3)(2,9)(-2,9)(-7,4)(7,4)}
    - b. {(1,3)(-1,3)(7,14)(-7,15)(9,3)(-9,3)}
    - c. {(-3,6)(7,1)(2,8)(4,6)(-4,6)(-2,8)(-7,1)(3,-6)}
    - d. {(0,2)(1,2)(-1,2)(4,9)(-4,9)}
4. Do you recognize a property common to the relations graphed below? Give a mathematical description of this property.

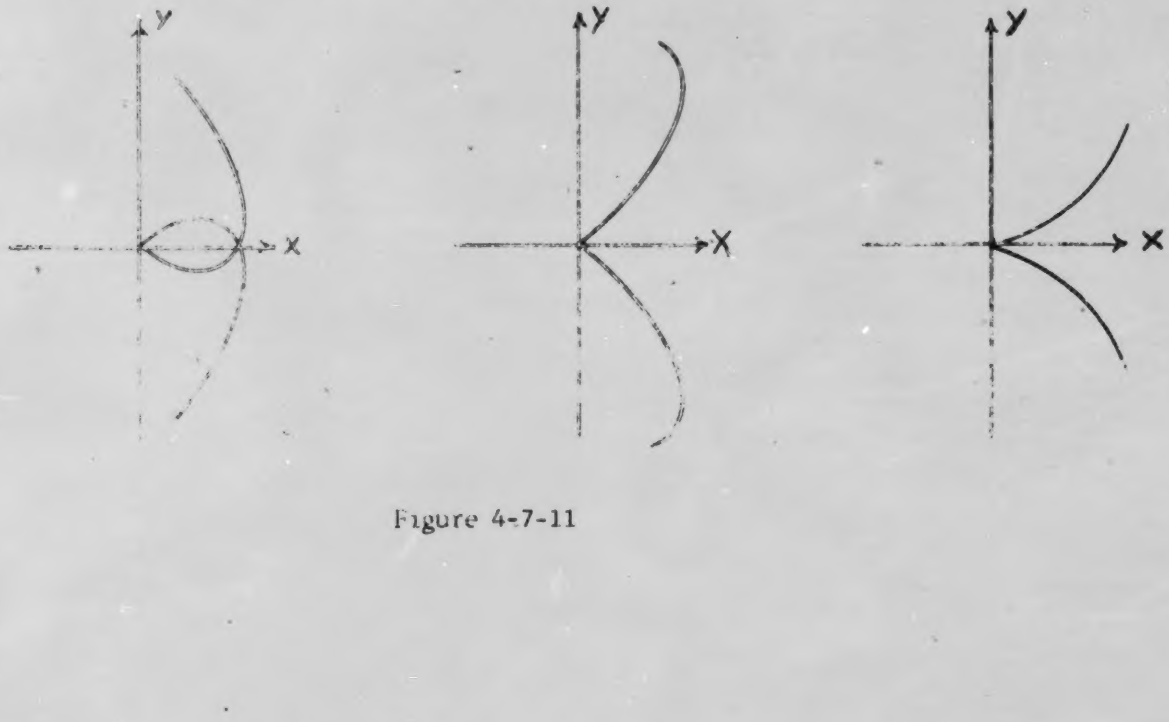


Figure 4-7-11

The graphs in Exercise 4-7-10 are illustrations of relations which have x-axis symmetry.

Definition 4-7-12   X–axis Symmetry

> A relation r is symmetric with respect to the x-axis if and only if for each $(a,b)\varepsilon r$, there exists $(a,-b)\varepsilon r$.

Example 4-7-13

$r = \{(1,2)(1,-2)(3,3)(3,-3)(4,1)(4,-1)\}$

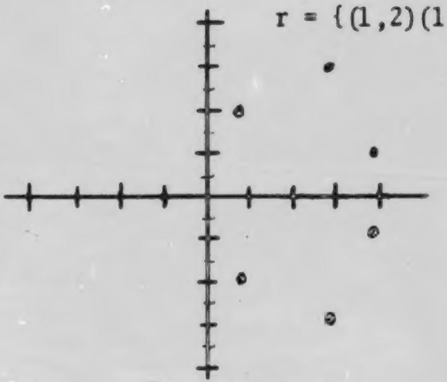


Figure 4-7-14

r is symmetric with respect to the x-axis.

Example 4-7-15

$s = \{(4,-1)(1,2)(4,1)(1,-2)(3,5)\}$

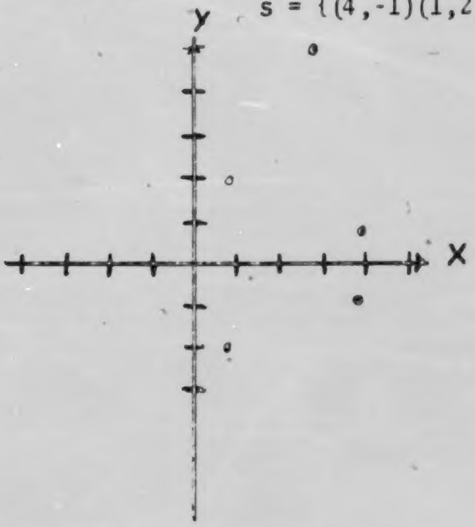


Figure 4-7-16

s is not symmetric with respect to the x-axis. Why?

Example 4-7-17

Prove that: $t = \{(x,y) | y^2 = x + 2\}$ has x-axis symmetry.

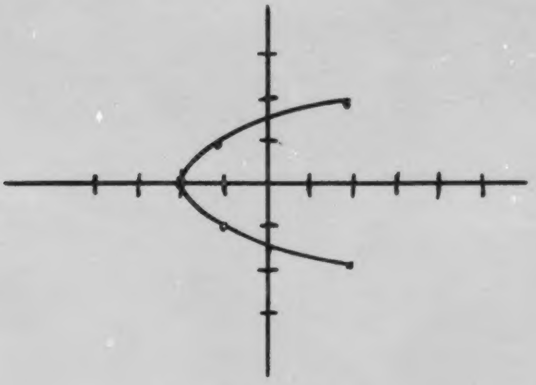


Figure 4-7-18

We must show that for each $(a,b) \epsilon t$ there exists $(a,-b) \epsilon t$.

Proof:

Given $t = \{(x,y) | y^2 = x + 2\}$

For any $(a,b) \epsilon t$

$b^2 = a + 2$	because (a,b) satisfies the set selector $y^2 = x + 2$
However $b^2 = (-b)^2$	$\forall x \quad x^2 = (-x)^2$
Therefore $(-b)^2 = a + 2$	by substitution.
Hence $(a,-b) \epsilon t$	because $(a,-b)$ also satisfies the set selector $y^2 = x + 2$

$\therefore$ For each $(a,b) \epsilon t$, there exists $(a,-b) \epsilon t$

Hence t has x-axis symmetry.

Exercise 4-7-19

1. Which of the following relations have x-axis symmetry?

   a. $\{(1,3)(2,9)(-7,4)(7,-4)(2,-9)(1,-3)\}$

   b. $\{(1,3)(4,7)(11,13)(-11,13)(-4,7)(-1,3)\}$

   c. $\{(1/2,0)(2,9)(2,-9)(-2,2)(-2,2)\}$

   d. $\{(x,y)|x = y^2\}$

   e. $\{(x,y)|y = 3x - 1\}$

   f. $\{(x,y)|x^2 + y^2 = 16\}$

   g. $\{(x,y)||x| + |y| = 6\}$

   h. $\{(x,y)|y = x^3\}$

2. Prove that the following relations have x-axis symmetry:

   a. $\{(x,y)|-y^2 = x + 2\}$

   b. $\{(x,y)|y^2 + x = 0\}$

3. Is it possible for a relation to be symmetric with respect to both the x-axis and the y-axis? If so, give an example.

4. Expand your computer program from Exercise4-7-10so it will check a finite relation for both x-axis and y-axis symmetry.

   Use your program to determine the symmetry properties for each of the following relations:

   a. $\{(1.29,3)(-1.29,3)(7,6)(13,11)(-13,11)(-7,6)\}$

   b. $\{(1,7)(2,9)(0,4)(-2,9)(-1,7)\}$

   c. $\{(14,-3)(7,22)(33,11)(-11,33)(-22,7)(3,14)\}$

   d. $\{(1,2)(-1,2)(1,-2)(-1,-2)\}$

   e. $\{(9,4)(9,-4)(2,1)(2,-1)(3,7)(3,-7)\}$

Consider the relations graphed below:

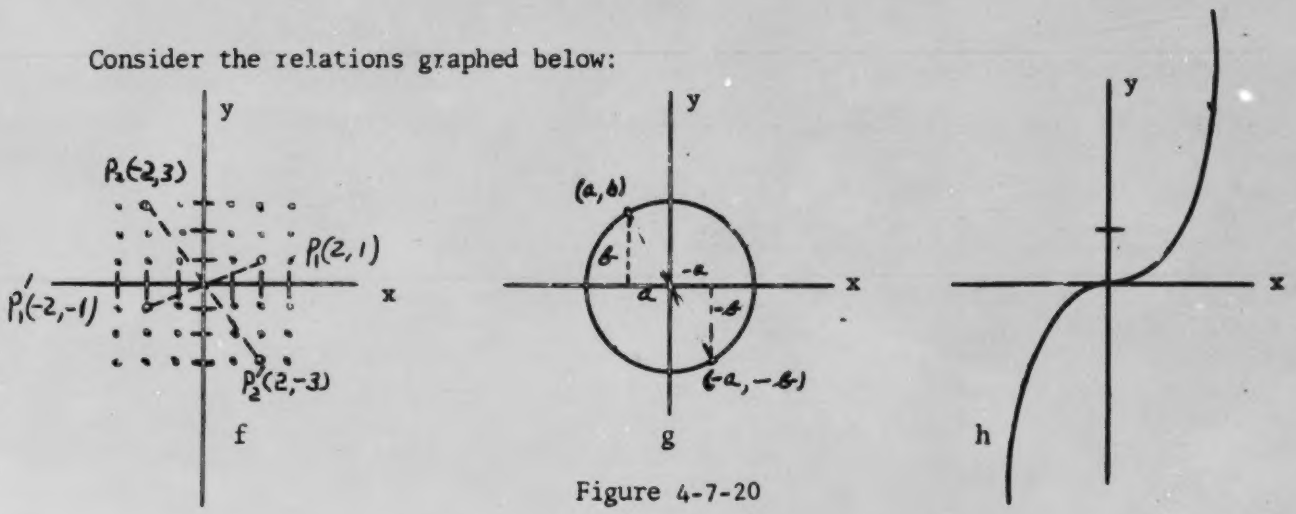


Figure 4-7-20

In graph of f we see that the relation contains the points $P_1(2,1)$, $P_1'(-2,-1)$ and $P_2(-2,3)$, $P_2'(2,-3)$. In fact, for each ordered pair (a,b) in f there exists an ordered pair (-a,-b) also in f.

In g, the end points of any diameter have coordinates such that the coordinates of one endpoint are additive inverses respectively of the coordinates of the other end point. That is if one end point has coordinates (a,b) then the other end point has coordinate (-a,-b).

The graph of relation h is a portion of the graph of a function that we will be dealing with later in this course. The graph suggests that h also has the property that for each ordered pair (a,b) in h, (-a,-b) is also in h.

Functions that have the property illustrated by f, g, and h above are said to be symmetric about the origin.

Exercise 4-7-21

1. Write a definition for symmetry about the origin.
2. Which of the following relations are symmetric about the origin?
    a. $\{(3,-1)(2,-2)(1,-3)(-3,1)(-2,2)(-1,3)\}$
    b. $\{(6,6)(-5,-1)(-3,-1)(5,1)(2,3)(3,1)\}$
    c. $\{(x,y)|y = x\}$
    d. $\{(x,y)|x^2 + y^2 = 1\}$

3. Prove that the following relations are or are not symmetric about the origin:

   a. $\{(x,y) | x^2 - y^2 = 3\}$

   b. $\{(x,y) | xy = -4\}$

   c. $\{(x,y) | y = x^3 - x\}$

4. Modify your program of Exercise 4-7-19 so that the computer will identify a given finite relation as symmetric about the x-axis, y-axis, both x-axis and y-axis, the origin, or none of these. Apply the program to the following relations:

   a. $\{(-7,1)(-6,2)(-5,3)(-4,4)(5,-3)(6,-2)(7,-1)\}$

   b. $\{(-7,1)(-6,2)(-5,3)(-4,4)(0,0)(7,1)(6,2)(5,3)(4,4)\}$

   c. $\{(-7,1)(-6,2)(-5,3)(-4,4)(-7,-1)(-6,-2)(-5,-3)(-4,-4)\}$

   d. $\{(-7,1)(-6,2)(-5,3)(-4,4)(-7,-1)(-6,2)(3,1)(-5,-3)(-4,-4)\}$

5. a. Prove or disprove that if a relation is symmetric about the x and y-axis then it is symmetric about the origin.

   b. Prove or disprove that if a relation is symmetric about the origin then it is symmetric about the x and y-axis.

4-8 Increasing and Decreasing Functions.

Let's consider Table 4-4-1 again with a column added. (See Table 4-8-1). Figures in the monthly cost column are arrived at by multiplying the number of school days in each month by the daily operating cost of $250.

Monthly Cost of Operating a School Building

Month	No. of School days/mo.	Monthly Cost
January	20	$5000
February	19	4750
March	23	5750
April	17	4250
May	20	5000
June	9	2250
July	0	0
August	0	0
September	19	4750
October	21	5250
November	19	4750
December	13	3250

Table 4-8-1

If we assign numerals 1 through 12 to the amounts in column three from the smallest amount to the largest, respectively, we have a function g consisting of the ordered pairs listed in Figure 4-8-2 Pairing the number of days in the second column with the corresponding amount in column three we have another function h as shown in Figure 4-8-3 g

(1,0)
(2,0)
(3,2250)
(4,3250)
(5,4250)
(6,4750)
(7,4750)
(8,4750)
(9,5000)
(10,5000)
(11,5250)
(12,5750)

Figure 4-8-2

h

(0,0)
(9,2250)
(13,250)
(17,4250)
(19,4750)
(20,5000)
(21,5250)
(23,5750)

Figure 4-8-3

Why do the two functions contain a different number of ordered pairs? Function h has only 8 ordered pairs because it is not customary to repeat elements in a set. For example, Table 4-8-1 contains the listing (19,4750) three times but we list it only once in the function h. In function g, there are 12 ordered pairs because we assigned a number to each amount in the third column of Table 4-8-1

Functions g and h have a property in common which is the subject of this unit. The property is that as the members of the domain increase the values of the function increase or remain constant. A function that has this property property is called an INCREASING function.

Example 4-8-4

$$j = \{(0,0),\ (1,1),\ (2,4),\ (3,9),\ (4,16)\ \ldots\ \}$$

The function j is an increasing function. Even though this function contains an infinite number of ordered pairs, it is still an increasing function because as the abscisses increase the value of the function increases.

Example 4-8-5

The function $f = \{(x,y) | y = [x]\}$ graphed below is also an increasing function.

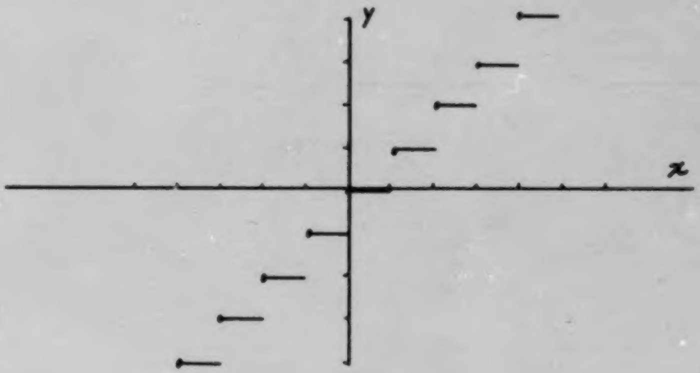


Figure 4-8-6

Definition 4-8-7

A function f is said to be an increasing function if and only if for every pair of numbers x_1 and x_2 in the domain of f such that $x_2 > x_1$, $f(x_2) \geq f(x_1)$.

Exercise 4-8-8

Which of the following functions are increasing?

a. $\{(6,15), (-2,0), (1,1), (8,22), (-1,1)\}$

b. $\{(0,0), (-1,-1), (1,1), (-2,-4), (2,2), (-3,-3), (3,3)\}$

c. $\{(x,y) | y = 2x - 3, -5 \leq x \leq 5, x \epsilon I\}$

d. $\{(x,y) | y = -3x - 1, -5 \geq x \geq 5, x \epsilon I\}$

e. $\{(x,y) | y = 3\}$

Although the functions g and h in Figure 4-8-2 have a property in common they are disimilar in another respect. The function values of g do not always increase but remain constant in some instances as the members in the domain increase. In function h the values always increase as the members in the domain increase. Functions which have the property that the function value always increases as the members of the domain increase are known as STRICTLY INCREASING functions.

Exercise 4-8-9

(a) Write the definition of a strictly increasing function.

(b) Which of the functions in Exercise 4-8-8 are strictly increasing?

Continuous functions may have the property of "increasing." Below are graphs of some continuous functions, which are "increasing functions."

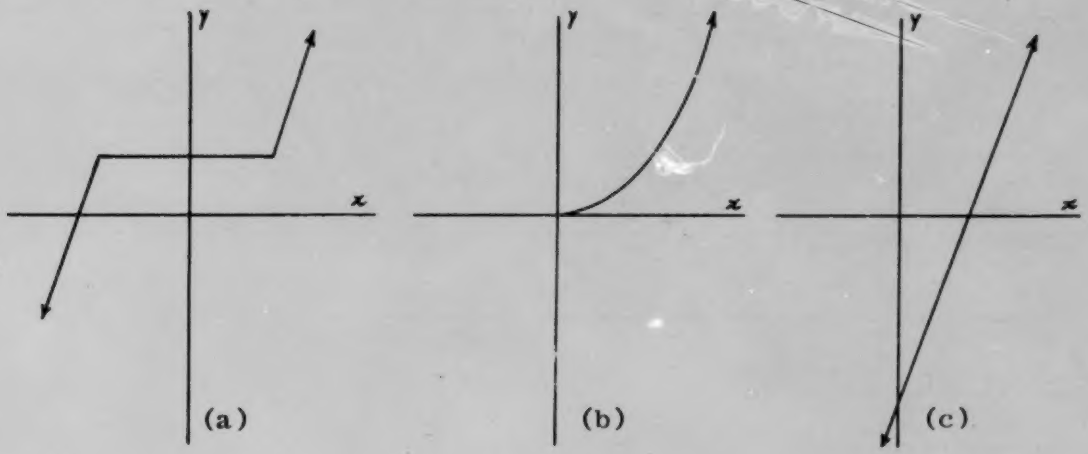


Figure 4-8-10

Let's look at the logic involved in determining if a finite function is increasing. We will assume that a finite function R containing N ordered pairs has been stored in the computer under the subscripted variables X(I) and Y(I).

R = {(X(1), Y(1)), (X(2)), . . (X(N), Y(N))}

The logic is quite simple The first step is to "sort" the ordered pairs according to their abscissas. This sorting should store the ordered pair with the smallest abscissa in the variables X(1) and Y(1). The abscissa should increase along with the subscripts and the ordered pair with the largest abscissa should be stored in the variables X(N) and Y(N). Once the function is "sorted" in this manner, it is a simple matter to determine if each successive ordinate is greater than or equal to the previous ordinate. A word of caution is in order concerning the sorting. Remember that whenever a pair of abscissas are interchanged during sorting, their ordinates must also be interchanged It might be interesting to see if you can find another algorithm to check for the increasing property without sorting. Can you?

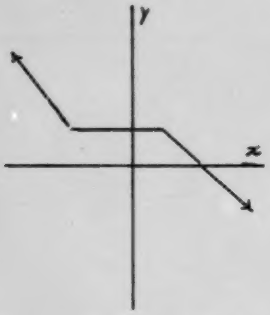
Exercise 4-8-11

(a) Write a computer program that will test a finite function to determine if it is increasing. Have the computer print out "increasing" or "not increasing" whichever applies.

(b) Apply the program to test functions (a) and (b) in Exercise 4-8-8

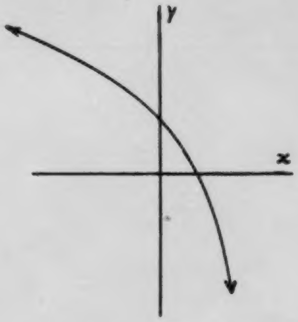
(c) Modify your program to test a finite function defined in set builder notation. Apply the program to function (c) and (d) of Exercise 4-8-8

Now we consider functions that are not increasing. The graphs below illustrate this type of function. The functions graphed in Figures 4-8-12a and 4-8-12b are called decreasing functions. The function graphed in Figure 4-8-12c is neither increasing nor decreasing.



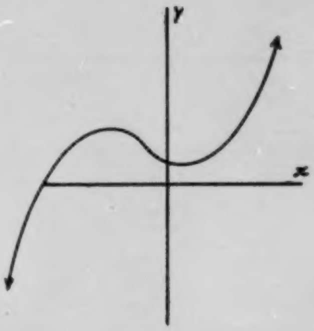
Decreasing

Figure 4-8-12a



Strictly Decreasing

Figure 4-8-12b



Neither Increasing nor decreasing

Figure 4-8-12c

Exercise 4-8-13

1. Write a definition of:

   a. Decreasing function

   b. Strictly decreasing function

2. Modify the computer programs written in Exercise 4-8-11 so that the computer will identify a finite function as "increasing" "decreasing" or "neither increasing or decreasing."

3. Apply your program to the following functions:

   1. $\{(0,0), (1,-1), (2,-2), (3,-3), (-1,1), (-2,2), (-3,3)\}$
   2. $\{(-5,8), (-4,6), (-3,6), (-2,4), (-1,2), (0,0)\}$
   3. $\{(2,5), (5,1), (8,-1), (-1,6), (-2,9)\}$
   4. $\{(-6,-11), (-3,-5), (9,19), (-14,-27), (0,1), (1,3), (-5,9)\}$
   5. $\{(x,y) | y = 6 - 2x, -5 \leq x \leq 5, x \epsilon I\}$
   6. $\{(x,y) | y = x^2 - x, -3 \leq x \leq 3, x \epsilon I\}$
   7. $\{(x,y) | y = x^2 - x, 0 \leq x \leq 5, x \epsilon I\}$

We have used the computer to find out if a finite function is increasing or decreasing. The computer could not be used to perform such a test on a continuous function containing an infinite number of ordered pairs. The computer simply cannot check every element in an infinite set. Consider the function in Figure 4-8-14.

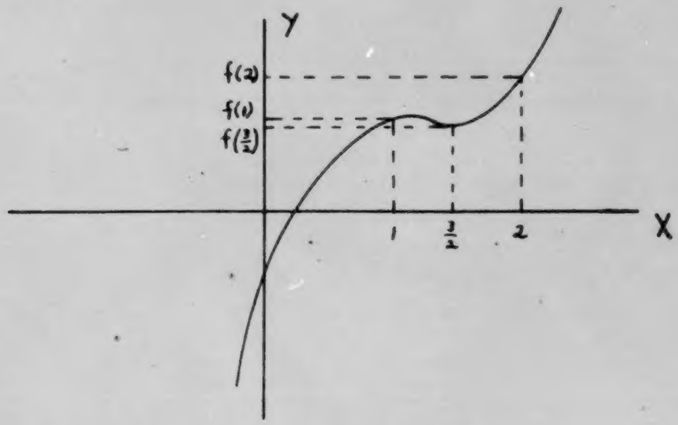


Figure 4-8-14

If we wanted to use the computer to test the function shown above for the increasing property, the best we could do would be to check over some finite subset of the domain. Assume we used the computer to check for the increasing property on $\{x | -100 \leq x \leq 100, x \in I\}$. This could be done by using a FOR-NEXT loop, stepping along from -100 to +100, evaluating the set selector at each integer.

Since $f(1) \leq f(2)$ and $1<2$ the computer would indicate the function was increasing. However, we can see from the graph that it is not. Notice that $f(3/2)<(1)$ while $3/2>1$. This is a contradiction of the definition which the computer would fail to detect in this check. One might argue that this failure could have been avoided by stepping along in increments of 0.5 or even 0.1. Although using the smaller increment would yield the proper result in this instance, there is no guarantee that these smaller intervals would by adequate for all other functions. It must be emphasized that checking a finite subset of an infinite function for a particular property and discovering that the property holds for that subset, is no guarantee that the property holds for the total function.

It is often useful to be able to determine the largest element in the range of a function, if it exists. Such an element is called the MAXIMUM value of the function.

Example 4-8-15

$f = \{(-9,26)(3,1)(9,14)(-13,7)(11,205)\}$

$f(11) = 205$ is the MAXIMUM value of the function f because it is larger than all other elements in the range.

Definition 4-8-16 MAXIMUM value of a function.

A function f with domain D has a MAXIMUM value $f(a)$, $a \in D$ if and only if for each $x \in D$, $f(a) \geq f(x)$.

Example 4-8-17

The function $g = \{(x,y) | y = 1/5(20x - x^2)\}$ has a MAXIMUM value $g(10) = 20$

The example above illustrates the fact that a function consisting of an infinite set of ordered pairs can have a MAXIMUM value. If you are not convinced of this, plot a graph of the function g.

Another example of an infinite function with a MAXIMUM value is shown in Figure 4-8-18

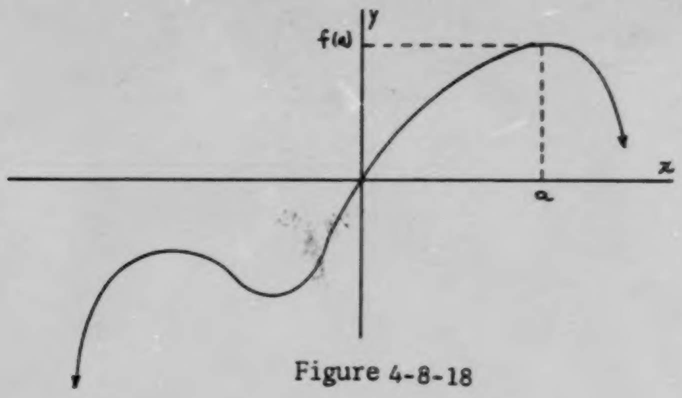


Figure 4-8-18

This function has a maximum value f(a) at x = a because for each x in the domain, $f(a) \geq f(x)$

Many functions, whether finite or infinite, do not have a maximum value. Such a function is shown in Figure 4-8-19 It does not have a maximum value because for any real number c there are function values greater than f(c).

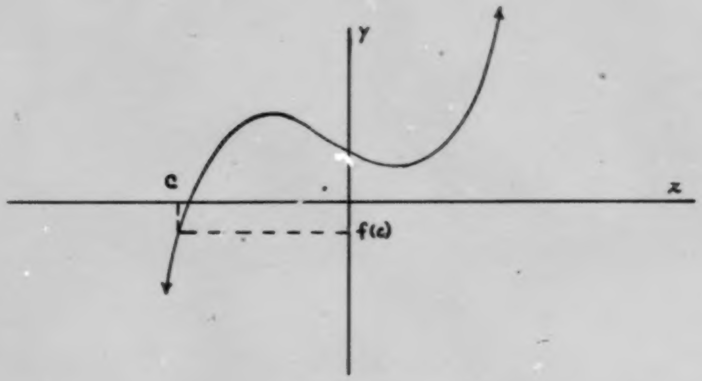


Figure 4-8-19

Other functions which may or may not have a maximum value often have a MINIMUM value. Such a function is shown in Figure 4-8-20 This function has a minimum value g(b) at x = b because g(b) is less than all other values of the function.

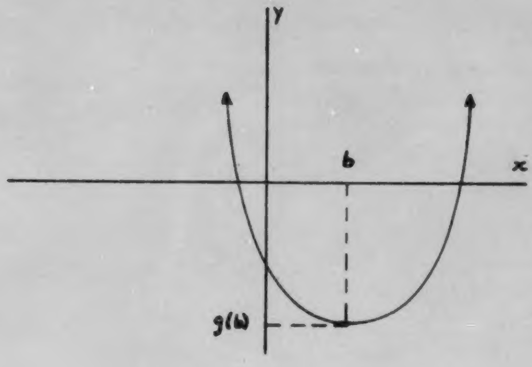


Figure 4-8-20

Exercise 4-8-21

1. Is it possible for a function to have a maximum value and a minimum value?
2. Write a mathematical definition for the "MINIMUM" value of a function.
3. Tell which of the following functions have (a) maximum value, (b) minimum value, (c) neither, (d) both.
   (1) $\{(2,1), (-1,7), (8,0), (1/2,16), (17,5)\}$
   (2) $\{(2,1), (-1,7), (8,0), (1/2,16), (-7,16)\}$
   (3) $\{(x,y) | y = x - 7\}$
   (4) $\{(x,y) | y = x^2 + 2\}$
   (5) $\{(x,y) | y = -x^2\}$
   (6) $\{(x,y) | y = x^3 - 4x\}$
4. Build a computer model of an algorithm for determining which of the following properties hold in a finite relation.
   (a) "increasing"
   (b) "decreasing"
   (c) "neither increasing or decreasing"
   (d) "has a maximum value"
   (e) "has a minimum value"
   (f) "has a maximum value and a minimum value"
   (g) "has neither a maximum value nor a minimum value"

5. Test your program on the following functions:

   (a) $\{(2,0), (1,-1), (3,-1), (7,5), (-2,-6)\}$

   (b) $\{(7,1), (8,2), (9,4), (10,6), (7\frac{1}{2} -2), (8\frac{1}{2},-3)$

   (c) $\{(x,y)\,|\,y = 4 - x^2, -5 \le x\varepsilon I \le 5\}$

   (d) $\{(x,y)\ y = \sqrt{54 - 3x^2}$ or $y = -\sqrt{54 - 3x^2}, -3 \le x \le 3, x\varepsilon I\}$

*6. Write a program that would test for maximum and minimum values of an infinite function defined by an equation over a given interval using certain increments within the given interval. Test your program on:

   (a) $\{(x,y)\,|\,y = x^3 - x, 0 \le x \le 1\}$

   (b) $\{(x,y)\,|\,y = x^3 - x, -1 \le x \le 0\}$

   (c) $\{(x,y)\,|\,x^2 + y^2 = 9, x \ge 0\}$

   (d) $\{(x,y)\,|\,x^2 + y^2 = 9, -3 \le x \le 3\}$

* Optional.

4- 9 Periodic Functions

We will introduce the concept of periodic functions with the following example. A science class was required to determine the percent of the moon in shadow for a period of at least 100 days following a full moon. It was decided to estimate the shadow to the nearest 5 percent. The table below shows typical data collected.

Date	Percent Shadow	Date	Percent Shadow	Date	Percent Shadow
Feb. 2	0	March 7	40	April 17	75
Feb. 7	25	March 9	60	April 21	25
Feb. 10	65	March 13	85	April 26	0
Feb. 13	85	March 17	95	May 1	25
Feb. 16	100	March 22	55	May 5	65
Feb. 19	90	March 26	15	May 10	100
Feb. 22	75	March 30	5	May 15	80
Feb. 25	30	April 3	30	May 19	30
March 1	0	April 7	65	May 23	5
March 4	20	April 12	100	May 27	20

Table 4-9-1

From this table can you predict the date of the next full moon following the last day of observation?

The students were able to sketch the following graph of their data.

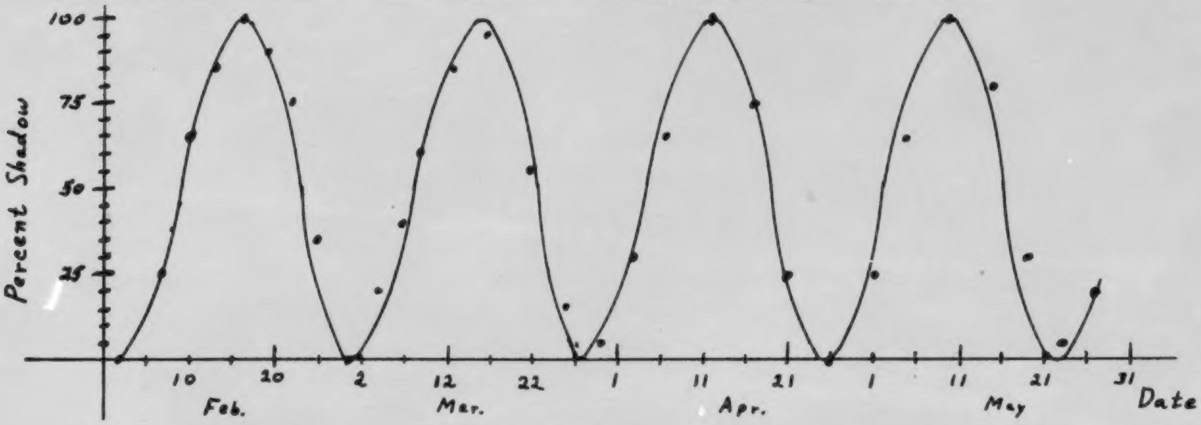


Figure 4-9-2

We can see from Figure 4-9-2 that the graph of the data is a simple curve which is repeated over and over. For example, the portion of the graph between March 1st and March 28th appears to be identical to the portion of the graph between February 2nd and March 1st. In addition, we can see that a given value for the percent shadow repeats in a regular manner. The moon is 50% in shadow on February 7th, March 8th, and April 5th. Notice that this repetition occurs at an interval of 28 or 29 days.

Assuming a regular motion for the moon and recognizing that student's estimates of the percent shadow are not exact, we might propose that this function is one which repeats indefinitely in cycles, each cycle being approximately 28 days in length. Such a function is said to be periodic.

Exercise 4-9-3

1. Approximately how many days must follow a day with 75% shadow before a day with 30% shadow occurs?
2. Try to write a concise definition of a periodic function.

Many physical phenomena are periodic in a manner similar to the example given above. Mathematicians have developed a very important class of functions called "periodic functions" to describe such occurrences. We will begin our discussion with the definition of such functions.

Definition 4-9-4 Periodic Function

A function f is periodic with period p, $p \varepsilon R$, $p \neq 0$ if and only if for all x in the domain of f, $f(x) = f(x + p)$.

Example 4-9-5

$$f = \{(x,y) \mid y = x - 5 \llbracket x/5 \rrbracket, x \varepsilon I\}$$

The function is partially described below:

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
f(x)	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1

a. Determine if the function is periodic and give the value of the period p.

Note the repetitious nature of the function f.

f(-5)f= f(0)=f(5)=f(10)=f(15)=f(20)= ...=0

f(-4)f= f(1)=f(6)=f(11)=f(16)=f(21)= ...=1

f(-3)f= f(2)=f(7)=f(12)=f(17) ... =2

and so on.

Hence the function is periodic and a value for the period is p = 5. The periodicty of f is readily apparent in the graph below.

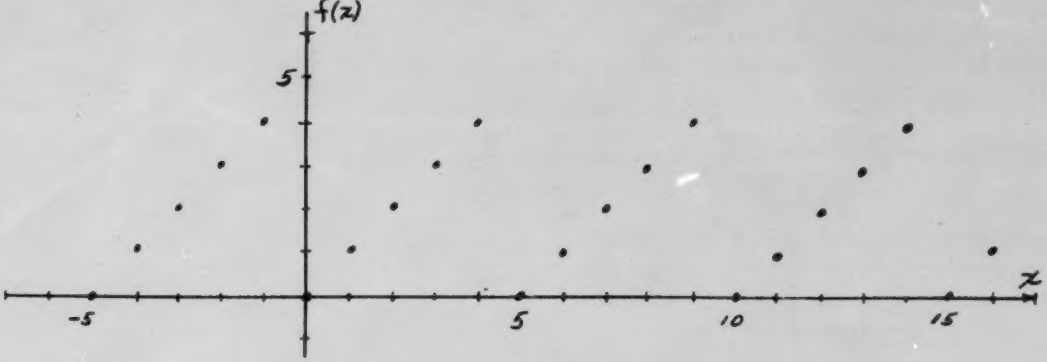


Figure 4-9-6

b. Are there values of p other than 5 which will satisfy the definition of periodicity?

Note that

f(5)=f(15)=f(25)= = 0

f(6)=f(16)=f(26)= = 1

Hence the function is periodic with period p = 10 or p = 2(5). It is also periodic for periods of 15, 20, 25 or for any value n·5 where n is an integer.

In Example 4-9-5 we looked at a function which was periodic for many values of p. We also recognized something unique about the number 5 in this particular example, namely that all of the periods, p, were integral multiples of 5. The number 5 is called the fundamental period of the function f because it is the smallest positive number p such that f(x)=f(x + p).

In addition to discovering the fundamental period of the function f, in Example 4-9-5 we also discovered that the function f is periodic for any period which is an integral multiple of the fundamental period. In general, a function which has fundamental period p, will also be periodic for periods $n \cdot p$ where n is an integer. In further study of mathematics you will be introduced to a method of proof known as Mathematical Induction. This method can be used to prove the generalization above.

Example 4-9-7

If $g = \{(x,y) | y = x - [\![x]\!]\}$, determine if the function is periodic and, if so, find its fundamental period.

Since a graph would aid us in answering this question we will use the computer to generate an $(x, g(x))$ table. We will then use this table to graph the function. A computer program for generating this table along with the print-out from the computer is shown below.

```
EXAMP2      16:09    C.S.S.   WED. 02/26/69


10  PRINT "X",TAB(10);"G(X)"
20  PRINT
30  FØR X=0 TØ 5 STEP 0.25
40  PRINT X;TAB(10);X-INT(X)
50  NEXT X
60  END

READY


RUN
```

```
EXAMP2     16:09   C.S S.  WED.02/26/69

X          G(X)

0          0
.25        .25
.5         .5
.75        .75
1          0
1.25       .25
1.5        .5
1.75       .75
2          0
2.25       .25
2.5        .5
2.75       .75
3          0
3.25       .25
3.5        .5
3.75       .75
4          0
4.25       .25
4.5        .5
4.75       .75
5          0
```

Figure 4-9-8

A graph of the function g is shown in Figure 4-9-9

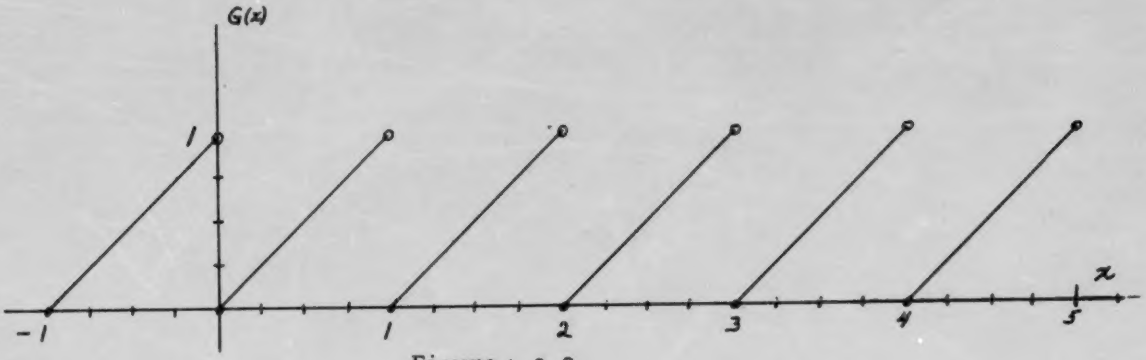


Figure 4-9-9

We can see that g is periodic because for each x in the domain of g, $g(x) = g(x + 1)$. Since the number one is the smallest positive number for which the function is periodic, the fundamental period is one. This function will also be periodic for periods 19, -37, 2, and 43. Why?

Exercise 4-9-10

1. Is it possible for a finite function to be periodic under our definition of a periodic function?

2. For each of the following functions, determine if the function is periodic. If it is, give the fundamental period. If it is not, show a counter example to the definition.

a. $f = \{(0,3)(1,4)(2,5)(4,0)(5,1)(3,3)\}$

b. $g = \{(1,2)(2,3)(3,4)(4,2)(5,3)(6,4)(7,2)(8,3)(9,4)\}$

c. $h = \{(x,y) | y = 1 \text{ if } x \text{ is even, } y = 0 \text{ is } x \text{ is odd, } x \epsilon I\}$

d. $j = \{(x,y) | y = x - 3\}$

e. $k = \{(x,y) | y = x - 2 [x/2], x \epsilon I\}$

f. $m = \{(x,y) | y = [x] - x\}$

3. Run the program shown in Example 4-9-7 using each of the following STEPS in the FOR-NEXT loop.

i 0.2

ii 0.125

iii 0.1

iv 0.0625

Careful analysis of the four different print-outs from this program reveals an inconsistency.

For example:

$g(2) = 0$ for STEP 0.125 or STEP 0.0625

but $g(2) = 1$ for STEP 0.2 or STEP 0.1

a. Decide which pair of print-outs is accurate and which pair is inaccurate.

b. Explain this inaccuracy on the part of the computer.

Hint: Express each STEP value as the ratio of two integers.

CHAPTER 5

Linear Equations and Inequalities

5-1 Introduction

We have seen in Chapter four that relations are sets of ordered pairs and that these ordered pairs may be shown as a graph on the Cartesian plane. In this chapter we will show that any linear arrangement of points on the Cartesian plane represents the graph of a relation of the form $\{(x,y)|Ax + By + C = 0, A \neq 0 \text{ or } B \neq 0\}$. Conversely, it will be shown that any relation of the form $\{(x,y)|Ax + By + C = 0, A \neq 0 \text{ or } B \neq 0\}$, is a straight line when plotted on the Cartesian plane. Solution sets of systems of two or more linear relations will also be investigated.

In many of the exercises in this chapter you will be asked to write computer programs. These programs will be used to build a computer solution to a comprehensive problem involving all of the principles studied in this chapter. Each student should keep copies of all programs written to assist him with this comprehensive problem.

5-2 The Slope of a Line.

Given any two points in the Cartesian plane, their positions relative to each other are of interest. Since two points determine a line, the positions of these points will dictate all of the properties of the line. Two of these properties are the amount of inclination and the points of intersection with the axes.

Before we can discuss these properties we need the following definition.

5-2-1 Definition. Delta x, Δx, and Delta y, Δy.

Given two points in the Cartesian plane, P_1 (x_1,y_1) and $P_2(x_2,y_2)$:

$$\Delta x = x_2 - x_1$$

$$\Delta y = y_2 - y_1$$

(Δx is read "delta x")

Example: (a) Given $P_1(1,2)$ and $P_2(5,7)$

$$\Delta x = (5)-(1) = 4$$

$$\Delta y = (7)-(2) = 5$$

(b) Given $P_1(-8,3)$ and P_2 $(4,-6)$

$$\Delta x = (4)-(-8) = 12$$

$$\Delta y = (-6)-(3) = -9$$

Exercise 5-2-2

1. Complete the following table:

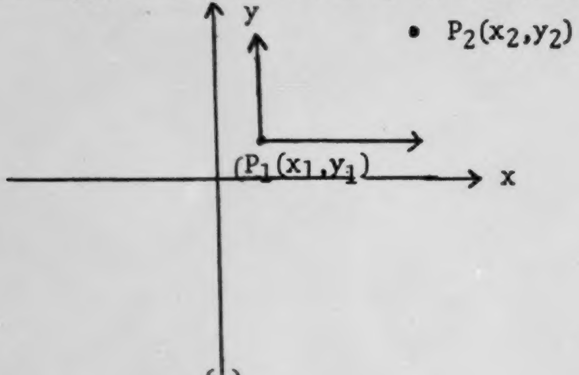
	P_1 (x_1,y_1)	P_2 (x_2,y_2)	Δx $x_2 - x_1$	Δy $y_2 - y_1$
a.	(3,4)	(5,0)	2	-4
b.	(5,0)	(3,4)	-2	
c.	(-3,4)	(3,6)		
d.	(3,6)	(-3,4)		
e.	(7,-5)	(-3,-6)		
f.	(-6,3)	(-6,-1)		
g.	(-6, -1)	(-6,3)		
h.	(2,9)	(9, 9)		
i.	(9, 9)	(2, 9)		
j.	(, 3)	(7, 9)	2	
k.	(7, -3)	(5,)		-4
l.	(,)	(5, 9)	4	8
m.	(-3, 2)	(,)	5/3	-5

2. Refer to table above. When the coordinates of P_1 and P_2 were interchanged how did this affect the sign of Δx and Δy?

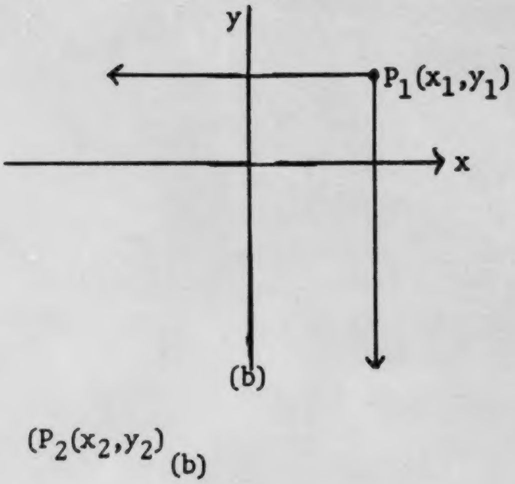
3. What generalization can you make concerning the ratio $\frac{\Delta y}{\Delta x}$ for all lines which slope down and to the left?

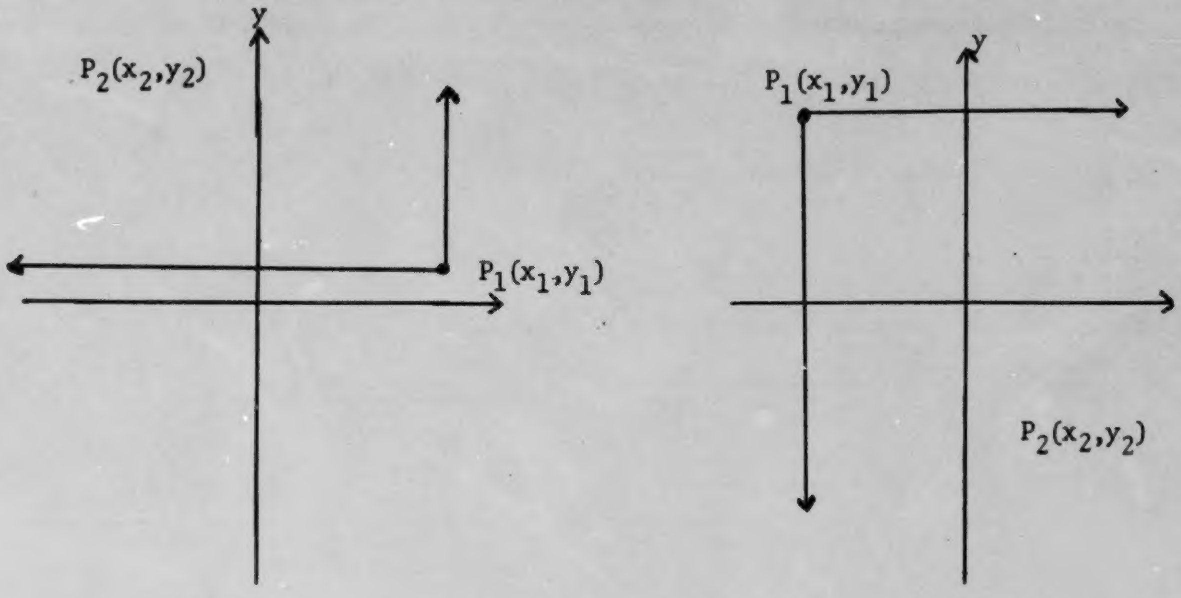
4. What happens to the ratio $\frac{\Delta y}{\Delta x}$ when points P_1 and P_2 are interchanged? (Exercise 5-2-2,1a,b,c,d)

The significance of Δx and Δy is illustrated in the following diagrams:



(a)





(c) (d)

Figure 5-2-3

In each case there are two points in the Cartesian plane, $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. An arrow is drawn from P_1, parallel to the x-axis. This horizontal arrow terminates at the line $x = x_2$. The arrow is a physical model of Δx. Its length represents the absolute value of Δx and its direction, the sign of Δx.

For example in Figure 5-2-3 (a),

$$x_2 > x_1$$

$$\therefore x_2 - x_1, > 0$$

$$\therefore \Delta x = x_2 - x_1 > 0$$

This is represented by the horizontal arrow to the right. Whenever $x_2 < x_1$, Δx is negative and is represented by an arrow to the left. (Figure 5-2-3 b,c)

In like manner an arrow can be drawn to illustrate Δy.

The significance of Δx is that it represents the change in first coordinates from P_1 to P_2. Similarly Δy is the change in second coordinates from P_1 to P_2.

We will now use Δx and Δy to discuss a property of the line determined by two points $P_1(x_1, y_1)$ and $P_2(x_2,y_2)$. The ratio $\frac{\Delta y}{\Delta x}$ is called the slope of the line. The slope is sometimes refered to as the inclination of the line and is a real number.

Definition 5-2-4 Slope of a Line.

Given two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ the slope, m, of the line containing these points is defined as:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \quad (x_1 \neq x_2)$$

If $x_1 = x_2$ the slope is undefined.

Example: The line containing points P_1 (1, -2) and P_2 (3, -8) has a slope of:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{\Delta y}{\Delta x}$$

$$m = \frac{(-8) - (-2)}{(3) - (1)}$$

$$m = \frac{-6}{2} \text{ or } -3$$

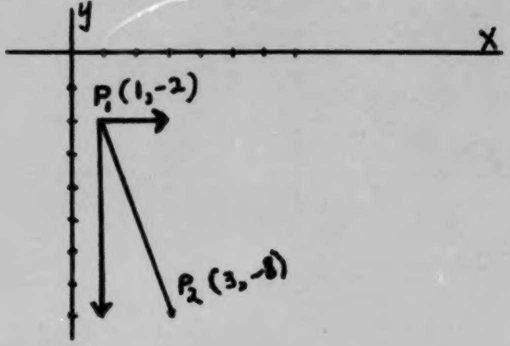


Figure 5-2-5

Exercise 5-2-6

1. Use the table in Exercise 5-2-2 to compute the slope of the lines joining P_1 and P_2.
2. Plot a graph of each line segment $\overline{P_1P_2}$ from exercise 5-2-2 a - m.

Observations.

1. We have observed lines which rise as we proceed to the right. These, we know from Chapter 4 . to be increasing functions. Their slopes are positive. If a line slopes downward to the right, or is a decreasing function, its slope will be negative, as we have seen in the previous example.
2. If a line is horizontal, then its slope is zero. This is true because $y_2 - y_1 = \Delta y = 0$.
3. The slope of a vertical line is undefined since for any two points on such a line $\Delta x = 0$.
4. If a line slants steeply, its slope has a large absolute value. If it rises slowly, its slope will have a relatively small absolute value.
5. If we are given two points on a line either point may be denoted as P_1 and the other as P_2.

Exercise 5-2-7

1. Write a computer program to compute the slope of the line segment joining any two points in the Cartesian Plane.

2. Use this program to compute the slope of the segment connecting the following pairs of points.

   a. A(3,2) and B(-1, -2)

   b. C(-1,-2) and D(3,2)

   c. E(-3,4) and F(-1,-2)

   d. G(-4,-2) and H(-4,3)

   e. I(-5.5,2) and J(3.5,2)

   f. K(0,0) and L(3,-2)

   g. M(53.1,-.6) and N(-27.9,.73)

   h. P(1,$-\sqrt{3}$) and Q(2,$-2\sqrt{3}$)

   i. T(-1.32,-2.87) and (.73,93.7)

   j. R(1,k) and S(2,3k) (Slope only.)

3. If the computer failed to indicate the slope of each segment, make the proper changes in your program to correct this failure.

4. Explain the problems that were encountered in using the machine to calculate the slopes in problems 2h and 2j.

5. Prove:

$$\forall x_1, \forall x_2, \forall y_1, \forall y_2 \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Hint: Recall the theorem:

$$\forall x \forall y \ \frac{x}{y} = \frac{-x}{-y}$$

It is important to realize at this point that the slope of a line is constant for any pair of points on that line. This idea is demonstrated in Figure 5-2-9. We should also realize that given any three points, they are collinear it the slopes of the segments joining any two of them are equal. In the following theorem the notation $m\,\overline{P_iP_j}$ represents the slopes of the line segment determined by points P_i and P_j

Theorem 5-2-8

P_1 (x_1,y_1), P_2 (x_2,y_2) and P_3 (x_3,y_3), $x_1 \neq x_2$ and $x_2 \neq x_3$ and $x_1 \neq x_3$ are collinear if and only if $m_{\overline{P_1P_2}} = m_{\overline{P_2P_3}} = m_{\overline{P_1P_3}}$

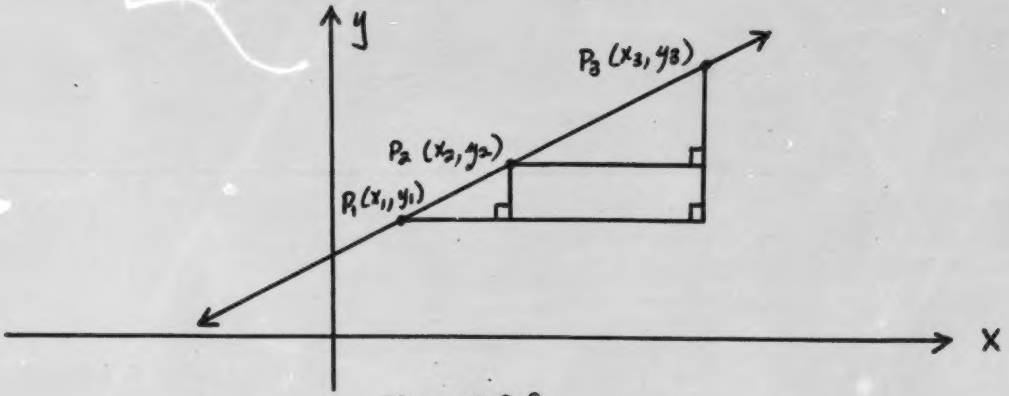


Figure 5-2-9

The proof of this theorem is left for the student. It follows from theorems on similar triangles. The contrapositive of this theorem will be useful in showing that if the slopes of the segments joining any two pairs of a triple of points are not equal then those points are not on the same line.

Theorem 5-2-10 (Contrapositive of Theorem 5-2-8)

$m\,\overline{P_1P_2} \neq m\,\overline{P_2P_3}$ or $m\,\overline{P_2P_3} \neq m\,\overline{P_1P_3}$ or $m\,\overline{P_1P_2} \neq m\,\overline{P_1P_3}$ if and only if

$P_1(x_1,y_1)$, $P_2(x_2,Y_2)$ and P_3 (x_3,y_3), $x_1 \neq x_2$ and $x_2 \neq x_3$ and $x_1 \neq x_3$, are non-collinear.

Exercise 5-2-11

1. Write a computer program which will determine whether or not 3 given points are collinear.

2. Use this program to determine whether or not the following triples of points are collinear.

   (a) A(-3,-2), B(-0,0), and C(6,4)
   (b) D(2,-5), E(-5,-5), and F(11.5,-5)
   (c) G(-3,0), H(0,2), and I(4,-5)
   (d) J(2,0), K(2,0.5) and L(2,8.25)
   (e) M(-3,11), N(42, 371) and P(-33, -229)
   (f) R(7,-4), S(27,-63) and T(-53,176)

Problem Set 5-2-12

1. In figure 5-2-13 notice that the triangles are similar and that the slope of the hypotenuse of one is the same as that of the other.

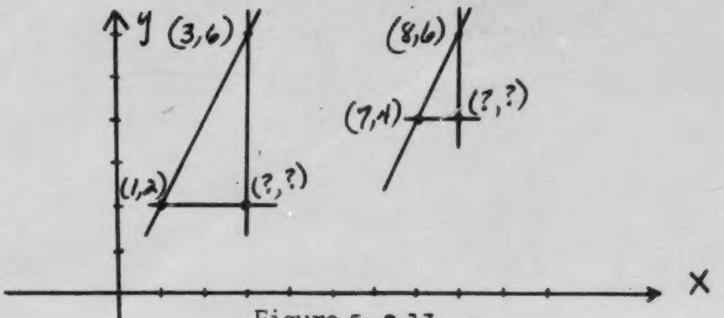


Figure 5-2-13

   a. Explain how you could determine that the four points are not collinear. Compare with the situation shown in figure 5-2-9.

   b. Construct a flow chart to illustrate how to differentiate between cases such as these.

2. Write a computer program which will compute the slopes of the line segments joining one selected point with each of the other points in the following sets and have the computer state whether or not the points are collinear.

   a. {(5,1), (5,-3), (5,7), (5,15)}

   b. {(-3,5), (-3,13), (1,10), (1,17)}

In problems 3-5 the program written in Exercise 5-2-11 may be used.

3. Find the slope, if defined, of each side of the following polygons:

   a. Triangle ABC, given A(1,-2), B(-3,0), C(-1,-6)

   b. Quadrilateral DEFG, given D(2,3), E(8,2.5), F(9,-1), and G(3,-0.5)

   c. Quadrilateral HIJK, given H(-1,3), M(2,0), N(0,3), and P(-2,0).

4. The points A(1,0), B(2,1), C(3,4), and D(2,3) are the vertices of a parallelogram.

   a. Find the slope of line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$.

   b. Compare the slopes of $\overline{AB}$ and $\overline{CD}$.

   c. Compare the slopes of $\overline{DA}$ and $\overline{BC}$.

5. The points A(0,-2), B(6,6), C(2,9), and D(-4,1) are the vertices of a rectangle.

   a. Find the slope of line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$.

   b. Compare the slopes of $\overline{DA}$ and $\overline{AB}$.

   c. Compare the slopes of $\overline{DA}$ and $\overline{BC}$.

5-3 Equation of a Line.

When considering any linear array of points on a graph, the set made up of the coordinates of these points is a relation. This set is a function for all linear arrays except the case when the points are all graphically arranged as a vertical line. We will now consider relations whose graphs are linear arrays of points as solution sets of linear equations.

Consider any infinite set, which is a relation, that graphs as a straight line. There are two conditions that may exist. First, if the relation is not a function, the domain of the relation is a constant, C. This relation graphs as a vertical line. (See Figure5-3-1).

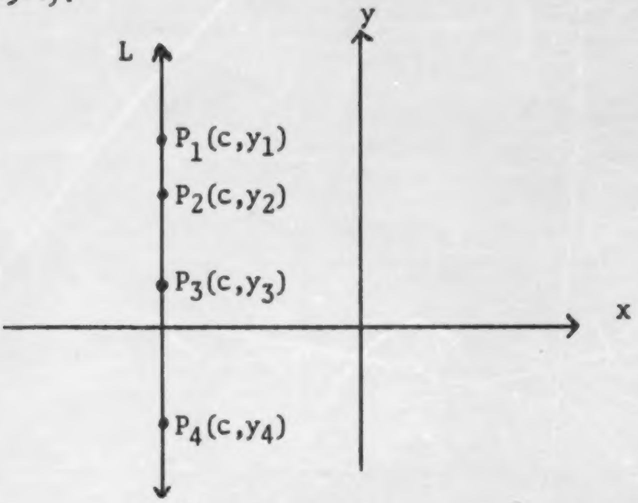


Figure 5-3-1

The relation $V = \{(x,y) : x = c\}$ is the set of all coordinates of points on this vertical, straight line. The equation $x = c$ is called the equation of the line.

Point - Slope Form of the Equation of a Line.

Consider any non-vertical line L with slope m. Assume point $P_1(x_1,y_1)$ lies on that line. (See Figure5-3-2)

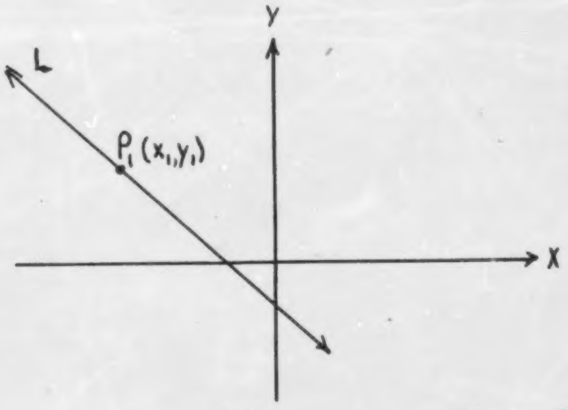


Figure 5-3-2

The function $G = \{(x,y): \frac{y - y_1}{x - x_1} = m, x \neq x_1\}$ contains coordinates of all points on L except P_1. This is the case because of the definition of slope. It tells us that the coordinates of any point on line L other than P_1 satisfy the equation $\frac{y - y_1}{x - x_1} = m$. Theorem 5-2-10 tells us that the coordinates of any point not on line L will not satisfy the equation $\frac{y - y_1}{x - x_1} = m$.

For example:

Consider the line L containing the point (-3,4) having slope of -3.

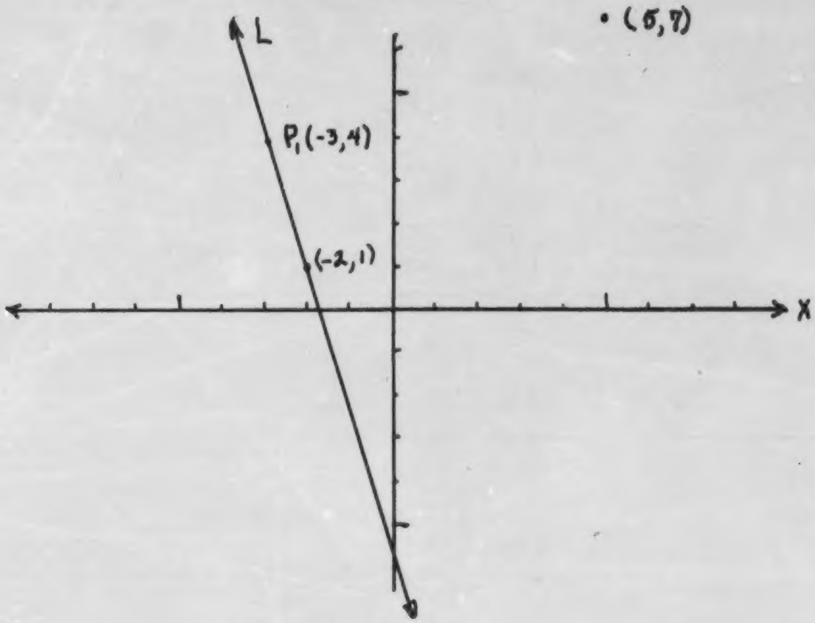


Figure 5-3-3

The function $G = \{(x,y): \frac{y - 4}{x - (-3)} = -3, x \neq -3\}$ contains the coordinates of all points in line L except P_1. The point (-2,1) is on L and belongs to this function because $\frac{1 - 4}{-2 - (-3)} = \frac{-3}{1} = -3$. The converse of Theorem 5-2-10 tells us that the point (5,7) is not on line L and does not belong to the function G because $\frac{7 - 4}{5 - (-3)} = \frac{3}{8} \neq -3$. The equation $\frac{y - y_1}{x - x_1} = m$ is called the point-slope form of the equation of a line L with slope m and containing point $P_1(x_1,y_1)$. It is used to find the equation of a line when you are given a point on the line and its slope.

Exercise 5-3-4

1. Construct a computer program which will print out the point-slope form of the equation of a line L given its slope and a point contained in the line.

2. In each of the following, write an equation of the line and draw its graph when:

   a. The slope is 0 and it contains the point A(4,0).

   b. The slope is -1 and it contains the point B(-8,-6).

   c. The slope is 2/3 and it contains the point C(-4,3).

   d. The slope is -3/2 and it contains the point C(-4,3).

   e. The slope is -3/2 and it contains the point D(4,3).

   f. It is parallel to the y-axis and it contains the point E(4,3).

3. Write a computer program to print the point-slope form of an equation of a line given two points on that line.

4. In each of the following find the slope, if it exists, and write the point-slope equation of the line containing the two points:

   a. A(1,2) and B(2,4)

   b. C(-3,-1) and D(2,-1)

   c. E(-1,2) and F(0,5)

   d. G(8,0) and H(8,4)

   e. I(4,4) and J(-2,-2)

   f. K(-3,3) and L(2,-2)

Problem Set 5-3-5

1. a. Does the line containing the points (4,-5) and (-6,7) also contain the point (-1,1)?

   b. Are the points (1,-1), (-2,-3) and (4,1) on the same line?

   c. Are the points (4,2), (-4,0), and (-8,-1) collinear?

2. Using the same set of axes, draw graphs of:

   a. $\frac{y-2}{x+2} = \frac{2}{3}$

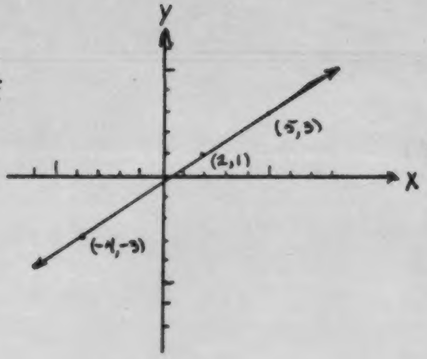
   b. $3(y - 2) = 2(x + 2)$

   c. $2x - 3y + 10 = 0$

   d. $y = \frac{2}{3}x + \frac{10}{3}$

   What do you observe about the four graphs?

3. Using the graph shown at the right, select two points, find the slope of the line, and write an equation of the line.



4. Find the equation of the line having an x-intercept,3, and y-intercept, -2. (The x-intercept is a point where the line crosses the x-axis.)

5. Write equations of the lines containing the sides of each of the following:

   a. Triangle ABC, given A(1,-2), B(-3,0), C(-1,-6)

   b. Quadrilateral KLMN, given K(-1,3), L(-2,-1), M(2,-2), N(3,2)

   c. Triangle DEF, given D(2,0), E(5,0), F(2,7)

   d. Quadrilateral PQRS, given P(2,0), Q(0,3), R(-2,0), S(0,-3)

   e. Write a computer program to solve 5a, b, c, and d.

Two-Point Form of the Equation.

If the coordinates of two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ on a non-vertical line, L, are known, it is possible to express the slope of the line; i.e.,

(1) $$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_2 \neq x_1$$

(2) We also know the function:

$$G = \{(x,y) \mid \frac{y - y_1}{x - x_1} = m, \; x \neq x_1\}$$

contains all ordered pairs which represent coordinates of points on the line L, except (x_1,y_1).

(3) By substituting (1) for m in (2) we obtain:

$$H = \{(x,y) \mid \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \; x \neq x_1 \neq x_2\}$$

The set selector in the relation H is known as the two-point form of the equation of a line. It should be noted that either of the two known points may be called P_1 or P_2.

Example:

Write the equation of the line determined by (1,4) and (3,-2) (See Figure 5-3-6). If we take the point (1,4) as P_1 and (3,-2) as P_2, it follows:

$$\frac{y - 4}{x - 1} = \frac{-2 - 4}{3 - 1}$$

This simplifies:

$$\frac{y - 4}{x - 1} = \frac{-6}{2}$$

$$2(y - 4) = -6(x - 1)$$

$$2y - 8 = -6x + 6$$

$$6x + 2y - 14 = 0$$

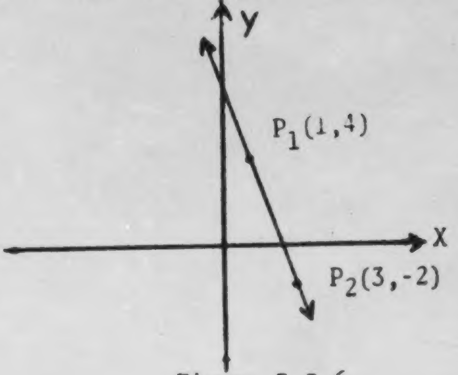


Figure 5-3-6

If (3,-2) had been selected as P_1, the results would be the same.

Exercise 5-3-7

1. Show that the same equation results in the above example, when (3,-2) is P_1 and (1,4) is P_2.
2. Write a program for the computer which will print out the equation of a line in two-point form, given the coordinates of two points of the line.
3. In each of the following, use the above program to print the equation of the line, in the two-point form.
    a. A(1,2) and B(2,4)
    b. C(-3,-1) and D(2,-1)
    c. E(-1,2) and F(0,5)
    d. G(8,0) and H(8,4)
    e. I(4,4) and J(-2,-2)
    f. K(-3,3) and L(2,-2)

The Slope-Intercept Equation of a Line

The point-slope equation of a line can be transformed to another equivalent equation, which is also useful, as follows:

$$* \quad \frac{y - y_1}{x - x_1} = m, \; x \neq x_1$$

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = mx - mx_1$$

$$y = mx - mx_1 + y_1$$

$$y = mx + (-mx_1 + y_1)$$

Since m, x_1, y_1 are elements of R, the expression $(-mx_1 + y_1)$ is an element of R. Let b represent $(-mx_1 + y_1)$. The equation becomes

$$y = mx + b$$

This equation is known as the slope - intercept form of a line. If we study the equation, we see that when $x = 0$, $y = b$. In other words, the line intersects the second axis at a distance of b units from the origin. If $b<0$ the line crosses the second axis below the origin and if $b>0$ the line crosses above the origin. The ordinate of the point at which the graph crosses the y-axis is known as the y-intercept of the line.

Example: Write the slope intercept equation of a line whose y-intercept is -5, and whose slope is 3/4 and draw the graph.

$$y = mx + b$$

$$y = \frac{3}{4}x + (-5)$$

To draw the graph of an equation in slope-intercept form, we first locate the y-intercept, and then draw the line through this point at the proper slope.

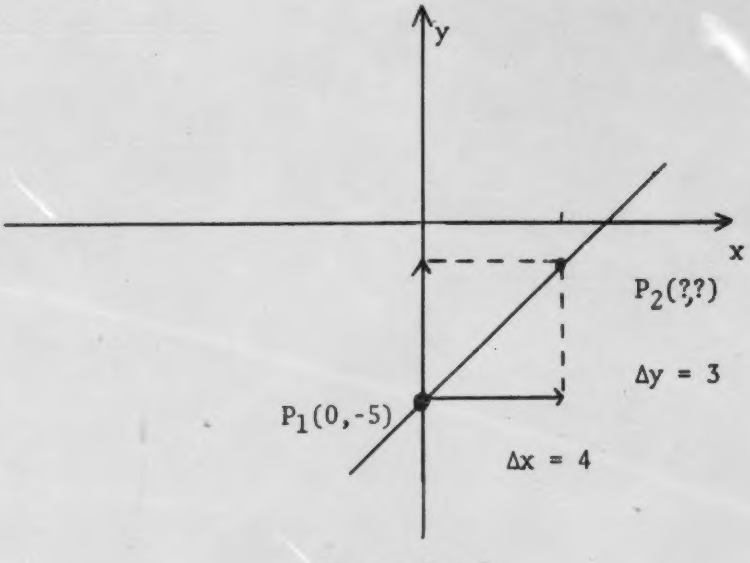


Figure 5-3-8

*All of these are forms of the equation of the described line.

Exercise 5-3-9

1. In each of the following write an equation of the line and draw its graph when:

   a. The slope is $-\frac{5}{4}$ and the y-intercept is -3.

   b. The slope is 4/5 and the y-intercept is -3.

   c. The slope is $-\frac{2}{3}$ and it contains the point (0,5).

   d. It contains the points (3,-6) and (0,4).

   e. The slope is 0 and the y-intercept is -4.

   f. The slope is 0 and the y-intercept is 0.

   g. The slope is not defined and it contains the point (-3,-3).

   h. The slope is not defined and it contains the point (0,5).

2. a. Write an equation of the line that is parallel to the x-axis and contains the point (2,-3).

   b. Write an equation of the line that is parallel to the y-axis and contains the point (2,-3).

3. Write a computer program which will print the slope-intercept form of the equation of the line containing the following pairs of points:

   A(3,2) and B(-1,-2)

   C(-3,4) and D(-1,-2)

   E(-4,-2) and F(-4,3)

   G(-5.5,2) and H(3.5,2)

   I(53.1,.-6) and J(-27.9,.73)

   K(-1.32,-2.87) and L(-73,93.7)

Problem Set 5-3-10

1. Write an equation for each of the lines containing the sides of each of the following:

a. Triangle DEF, given D(0,0), E(10,-4), F(2,5)

b. Quadrilateral GHIF, given G(2,3), H(8,2.5), I(9,-1), J(3,-0.5)

c. Quadrilateral ABCD, given A(-5,-2), B(0,-2), C(0,3), D(-5,3)

d. Will the program written in Exercise5-3-7, Number 2, produce the equations for the above cases? If not, modify it to do so.

2. Draw graphs of each of the following equations:

a. $y = 2x + 3$	c. $y = -5x - 3$	e. $y = -x - 1/2$
b. $y = -4x + 2$	d. $y = \frac{3}{2}x - 4$	f. $y = -\frac{5}{4}x + 2$

3. Write equations for the lines determined by each of the following sets of information.

a. A(-6,2) and B(1,-3)

b. $m = -\frac{2}{3}$, $b = 7$

c. C(0,1), $m = 5$

d. D(15,-7) and E(15,7)

e. m is undefined, F(0,5)

f. $m = -.625$, G(-.12,.237)

4. Write a "computer aided instruction" program which will ask a student a series of questions concerning the slope of a line; its x or y intercept; some point on the line or some other pertinent information. As soon as two pieces of information have been given, the computer should generate the equation in general form.

5-4 Graphs of Linear Functions

Having studied three forms for the equation of a line, point-slope, two-point, and slope-intercept, we now introduce the general form, $Ax + By + C = 0$, $A \neq 0$ or $B \neq 0$. It will be shown that each of the previous forms reduces to this "general form" and that this form is also of first degree. That is, each of the variables, x and y, is raised to the first power and there are no terms containing the product of these variables.

(1) Point-slope form:

Let $m = \frac{a}{b}$, $b \neq 0$, be the slope of a line containing the point $P_1(x_1,y_1)$.

Then $\frac{a}{b} = \frac{y - y_1}{x - x_1}$, $x \neq x_1$

$$a(x - x_1) = b(y - y_1)$$

$$ax - ax_1 = by - by_1$$

$$ax + (-b)y + (-ax_1 + by_1) = 0$$

This equation is now in the general form $Ax + By + C = 0$, $B \neq 0$ where $A = a$, $B = -b$, and $C = -ax_1 + by_1$.

(2) Two-point form:

Given a line containing two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, $x_1 \neq x_2$.

Then $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$, $x \neq x_1 \neq x_2$.

$$(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$$

$$yx_2 - y_1x_2 - yx_1 + y_1x_1 = xy_2 - x_1y_2 - xy_1 + x_1y_1$$

$$xy_2 - xy_1 + yx_1 - yx_2 + x_1y_1 - x_1y_2 + y_1x_2 - y_1x_1 = 0$$

$$(y_2 - y_1)x + (x_1 - x_2)y + (y_1x_2 - x_1y_2) = 0$$

This equation is also in the general form $Ax + By + C = 0$ where $A = (y_2 - y_1)$, $B = (x_1 - x_2)$, and $C = (y_1x_2 - x_1y_2)$, $(B \neq 0)$.

(3) Slope-intercept form

Let $m = \frac{a}{b}$, $b \neq 0$, be the slope of a line with y-intercept c.

Then $y = \frac{a}{b}x + c$

$by = ax + bc$

$ax + (-b)y + bc = 0$

This equation is now in the general form $Ax + By + C = 0$, $B \neq 0$, where $A = a$, $B = -b$, and $C = bc$.

We have now verified the fact that equations of nonvertical linear arrays of points on a Cartesian plane may be written in general form, $Ax + By + C = 0$, $B \neq 0$. The equations may also be expressed in the standard form, $Ax + By = C$, $B \neq 0$. We call functions defined by equations of these forms "linear functions." The vertical linear arrays of points can also be defined by equations of the form $Ax + By + C = 0$ if $B = 0$. Hence, we will refer to any relations of the form $\{(x,y) | Ax + By + C = 0, A \neq 0 \text{ or } B \neq 0\}$ as "linear relations."

We have shown that linear arrangements of points on a Cartesian plane represent relations of the form $\{(x,y) | Ax + By + C = 0, A \neq 0 \text{ or } B \neq 0\}$. We now want to prove that all relations of this form $\{(x,y) | Ax + By + C = 0, A \neq 0 \text{ or } B \neq 0\}$ are lines when graphed.

Theorem 5-4-1

The graph of a linear relation is a straight line.

Proof: Let $\{(x,y) | Ax + By + C = 0, A \neq 0 \text{ or } B \neq 0\}$ be a linear relation containing the elements (x_1,y_1), (x_2,y_2), and (x_3,y_3).

Case I: If $B \neq 0$ then by substitution of (x_1,y_1) into the set selector,

$$Ax_1 + By_1 + C = 0$$

we find

$$C = -Ax_1 - By_1$$

This means $Ax + By + (-Ax_1 - By_1) = 0$

(1) or $\frac{Y - Y_1}{X - X_1} = -\frac{A}{B}$

It is clear that equation (1) is equivalent to the set selector $Ax + By + C = 0$. Hence, (x_2,y_2) and (x_3,y_3) will satisfy equation (1).

i.e. $$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{A}{B}$$

and $$\frac{y_3 - y_1}{x_3 - x_1} = -\frac{A}{B}$$

by substitution, we get $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_3 - y_1}{x_3 - x_1} = -\dfrac{A}{B}$

From this, it can be shown algebraically that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

or $$m\,\overline{P_1P_2} = m\overline{P_2P_3} = m\overline{P_2P_3}$$

Therefore, by Theorem5-2-8 points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, and $P_3(x_3,y_3)$ are collinear and the graph of the relation must be a straight line.

Case II: If $B = 0$ and $A \neq 0$ the set selector becomes $Ax + 0y + C = 0$ which is equivalent to the equation $x = -\frac{C}{A}$. Since (x_1,y_1), (x_2,y_2), and (x_3,y_3) satisfy this equation, $x_1 = x_2 = x_3 = -\frac{C}{A}$. Consequently, points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_3(x_3,y_3)$ lie on the same vertical line.

Exercise 5-4-2

1. Graph each of the following and find the slope and y-intercept from the graph:

   a. $y - 3x - 2 = 0$

   b. $3x - 7y - 4 = 0$

   c. $2x + 4y = 8$

   d. $\frac{x}{5} + \frac{y}{2} = 1$

   e. $\frac{x}{3} - \frac{y}{4} = 1$

2. Write the slope-intercept equation equivalent to each of the following, and find the slope and y-intercept by inspection:

a. $3x + 2y = 7$

b. $\frac{x - 5}{y + 2} = \frac{3}{4}$

c. $\frac{2}{3}(x - 4) = \frac{3}{4}(y + 2)$

d. $\frac{x}{5} + \frac{y}{2} = 1$

e. $\frac{2}{3}(y - x) = 4 - \frac{2}{3}x$

f. $\frac{x}{a} + \frac{y}{b} = 1$

3. Write the slope-intercept equation of the line determined by each of the following pairs of points:

(a) (c,d,) and (0,0) b. (r,-3) and (2,s) c. (e,f) and (g,f)

4. Write the slope-intercept equation equivalent to $Ax + By + C = 0$

5. Solve for the slope and the x and y intercepts of the following equations.

a. $2x + 3y - 2 = 0$

b. $7x - 3y + 8 = 0$

c. $5x - 17y = 5$

d. $3x + 21y = -7$

e. $3x = 4y - 7$

f. $5x = 27y$

g. $3y = 39x + 18$

h. $y = -17x - 21$

i. $\frac{x}{3} + \frac{y}{5} + 30 = 0$

j. $\frac{4}{5}(y - x) = 3 - \frac{7}{4}x$

6. Show that if a line contains the points (a,0) and (0,b), where $a \neq 0$ and $b \neq 0$, then $\frac{x}{a} + \frac{y}{b} = 1$ is an equation of that line. This is known as the intercept form. Why?

7. Write an intercept form equation of the line which contains the points:

a. (3,0) and (0,2)

b. (-4,0) and (0,-5)

c. x - intercept is 2.736
y - intercept is -8.71

8. Draw lines which have the following properties:

a. The slope is 3 and the line contains the point P (0,0).

b. The slope is -2 and the line contains the point A (1,-1)

c. The slope is $-\frac{1}{3}$ and the line contains the point B (-3,2)

d. The slope is 0 and the line contains the point D (-3,-1)

e. The line is parallel to the first axis (the x-axis) and it contains E(-3,-1)

f. The line is parallel to the second axis (the y-axis) and contains F(2,-2)

9. Give the equation in "standard form" for each line in the previous exercise.

5-5 A Problem for Application

Having just reviewed the concepts of slope and linear equations, let's turn to a specific problem involving these ideas.

Great Goody

> Given the coordinates of any three points in the Cartesian plane, determine if they are collinear. If not, find the coordinates of the center of the circle containing these points and the length of its radius.

Do we have the mathematical knowledge and techniques to successfully attack such a problem? The problem stated in the first sentence of Great Goody has already been considered.

Exercise 5-5-1

1. Write a flow chart to illustrate the logical flow of ideas involved in solving Great Goody.
2. Construct a program which will completely handle the first sentence of Great Goody.

Read the last sentence of Great Goody again. The additional points of information we have to study to successfully complete the analysis of this problem are:

1. The calculation of the coordinates of the midpoint of a line segment.
2. Slopes of perpendicular and parallel lines.
3. Solution of a system of linear equations.
4. The distance between two points.

5-6 Dividing Line Segments

Given a segment whose endpoints are $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. We shall now derive a formula which can be used to compute the coordinates of the midpoint $M(x,y)$ of $\overline{P_1P_2}$ from the coordinates of P_1 and P_2.

First we will consider the case of a segment parallel to x-axis. When this has been done, we will use similar techniques to deal with the case of a segment parallel to the y-axis. From these two results, we will be able to derive the formula for the midpoint of any segment in the Cartesian plane.

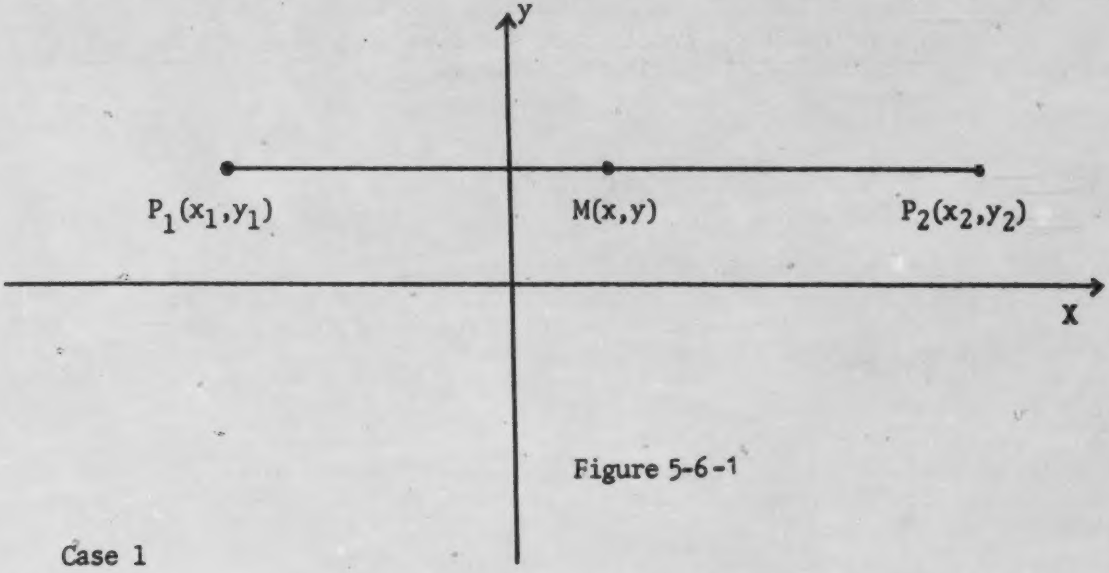


Figure 5-6-1

Case 1

Given points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ $x_1 \neq x_2$ such that the segment P_1P_2 is parallel to the x-axis.

By the definition of midpoint $M(x,y)$, is the only point on $\overline{P_1P_2}$ which is equidistant from P_1 and P_2.

Since $\overline{P_1P_2}$ is parallel to the x-axis, all points of $\overleftrightarrow{P_1P_2}$ have the same y-coordinate. Therefore $y_1 = y = y_2$. So the y-coordinate of the midpoint M is y_1.

Since $\overline{P_1P_2}$ is parallel to the x-axis, M is equidistant from P_1 and P_2 if and only if $|x_1 - x| = |x - x_2|$.

By the definition of absolute value this equation is equivalent to:

(1) $x_1 - x = x - x_2$ or

(2) $x - x_1 = x - x_2$

Since $x_1 \neq x_2$ equation (2) is false.

Consider equation (1)

$$x_1 - x = x - x_2$$

$$x_1 + x_2 = x + x$$

$$x = \frac{x_1 + x_2}{2}$$

This gives us the x_1 coordinate of the midpoint M. Thus, the midpoint M of the segment $\overline{P_1P_2}$ has coordinates $\left(\frac{(x_1 + x_2)}{2}, y_1\right)$.

Case 2.

In a similar manner, if $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, $y_1 \neq y_2$, are the end points of a segment parallel to the y-axis, then midpoint M will have coordinates $(x_1, \frac{y_1 + y_2}{2})$.

The third case, that of a segment which is parallel to neither axis, will now be discussed.

Case 3.

Given points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ such that

$x_1 \neq x_2$ and $y_1 \neq y_2$ and Point M the midpoint of $\overline{P_1P_2}$

Draw lines parallel to the x-axis through P_1 and M. Draw lines parallel to the y-axis through P_2 and M. Label the points of intersection as indicated in Figure 5-6-2

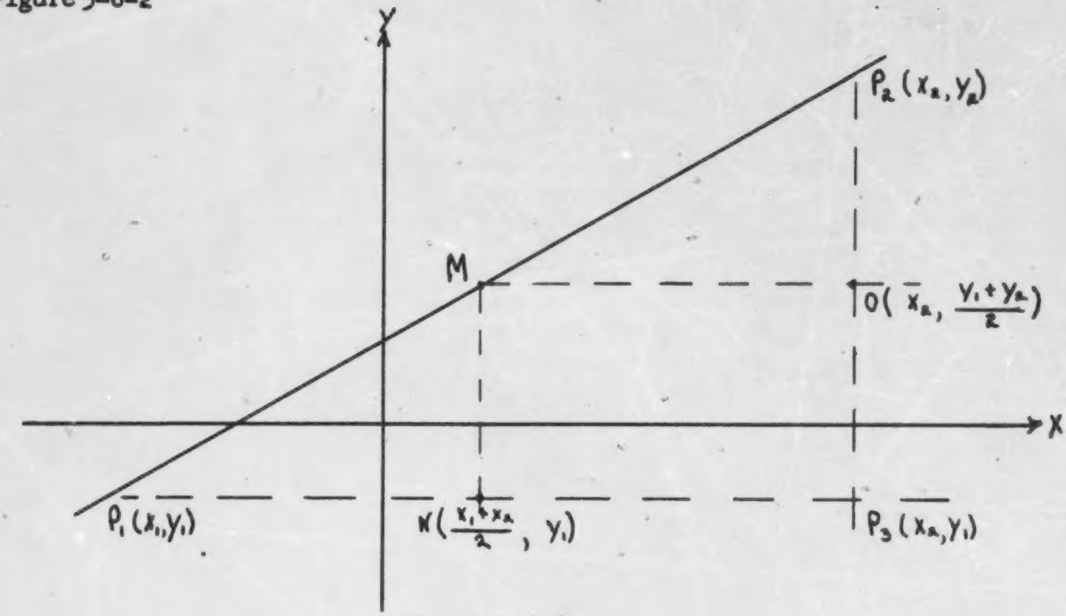


Figure 5-6-2

Since M is the midpoint of $\overline{P_1P_2}$ and $\overline{MO} \| \overline{P_1P_3}$, point O is the midpoint of $\overline{P_2P_3}$. From the previous discussion the coordinates of O are $(x_2, \frac{y_1 + y_2}{2})$

By similar reasoning N is the midpoint of $\overline{P_1P_3}$ and has coordinates ($\frac{x_1 + x_2}{2}$, y_1).

M will have the same x coordinate as N and the same y coordinate as 0, since they are on vertical and horizontal lines respectively.

Now let us state this as a theorem.

Theorem 5-6-3

The midpoint formula: $\forall P_1(x_1y_1), \forall P_2(x_2,y_2)_1$ $\forall M(Xm,Ym)$, M is the midpoint of $\overline{P_1P_2}$ if and only if

$$X_m = \frac{x_1 + x_2}{2}$$

$$Y_m = \frac{y_1 + y_2}{2}$$

Exercise Set 5-6-4

1. Write a computer program which will find the coordinates of the midpoint of a segment determined by any two points in the coordinate system.

2. Using the above program, find the coordinates of the midpoints of the segments defined by the following pairs of points:

   a. (1,3), (5,7)

   b. (4,1), (5,6)

   c. (-3,1) (-3,-2)

   d. (4,-1), (-7,-1)

   e. (-2.0786,5.173),(-99.99,-1.864)

   f. (3/2,9), (-6, 2/3)

   g. (r,s), (r,p)

3. Write a computer program which will determine the coordinates of one endpoint of a segment, given its midpoint and the other endpoint.

4. Using the following coordinates, the first pair being the coordinates of the midpoint and the second pair being one of the endpoints, find the coordinates of the other endpoint.

   a. (-1,7),(-3,5)

   b. (-1,5), (8,2)

   c. (-1,-7), (3,4)

   d. (-86.237,2.153),(1.002,-200.6)

   e. (-22.7,-8.32),(3/4,87/101)

   f. (c,r), (d,s)

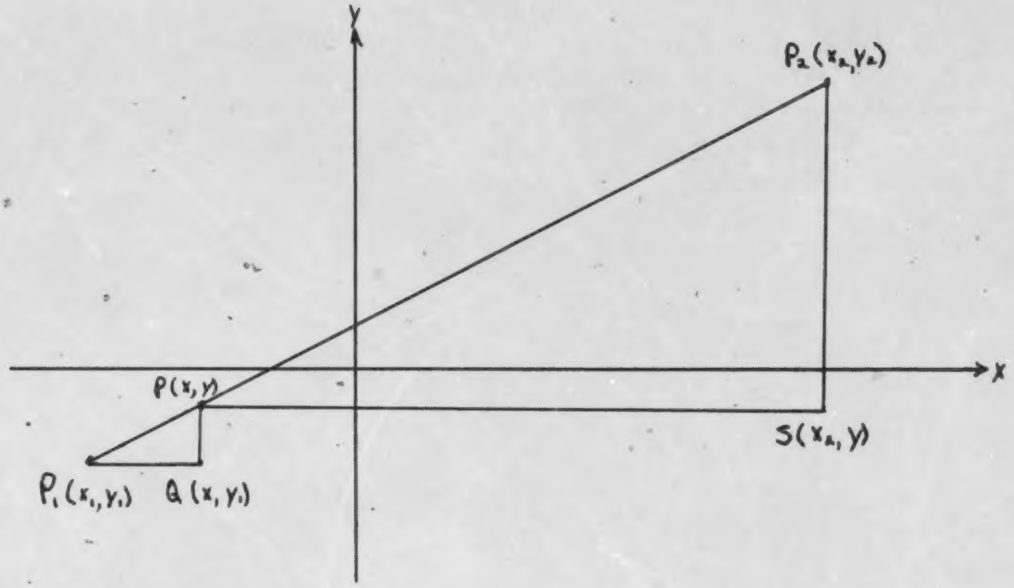
Problem Set 5-6-5

1. Given a triangle with vertices at A(9,-5), B(-5,7) and C(-4,-6). Write a program to compute midpoints of all three sides and slopes of all line segments in the figure.

2. The following pairs of points are endpoints of line segments. For each segment, find the coordinates of a point which divides it in the specified ratio.

a. (0,0), (0,5); 2:3	d. (2,3), (6,7); 1:3
b. (0,0), (0,5); 3:2	e. (-3,2), (4,-5); 2:5
c. (1,1), (1,7); 2:1	f. (-5,2), (6,7); 3:1

3. Derive formulas for the coordinates of a point that divides $\overline{P_1P_2}$ in the ratio $r_1:r_2$.



5-7 Parallel and Perpendicular Lines.

In the previous sections we observed that whenever two different lines are parallel they have the same slope and conversely if they have the same slope then they are parallel. We will now prove this generalization.

Theorem 5-7-1.

Different non-vertical lines have the same slopes if and only if they are parallel.

Part I.

If two lines have the same slope, then the lines are parallel.

Proof:

Figure 5-7-2 shows two lines L_1 and L_2 with the same slope. A horizontal line, L, has been drawn as a transversal. Perpendiculars from P_1 and P_2 to L have been drawn forming the triangles $P_1M_1A_1$ and $P_2M_2A_2$. Since the slopes of L_1 and L_2 are the same, we know that $\frac{\Delta y_1}{\Delta x_1} = \frac{\Delta y_2}{\Delta x_2}$.

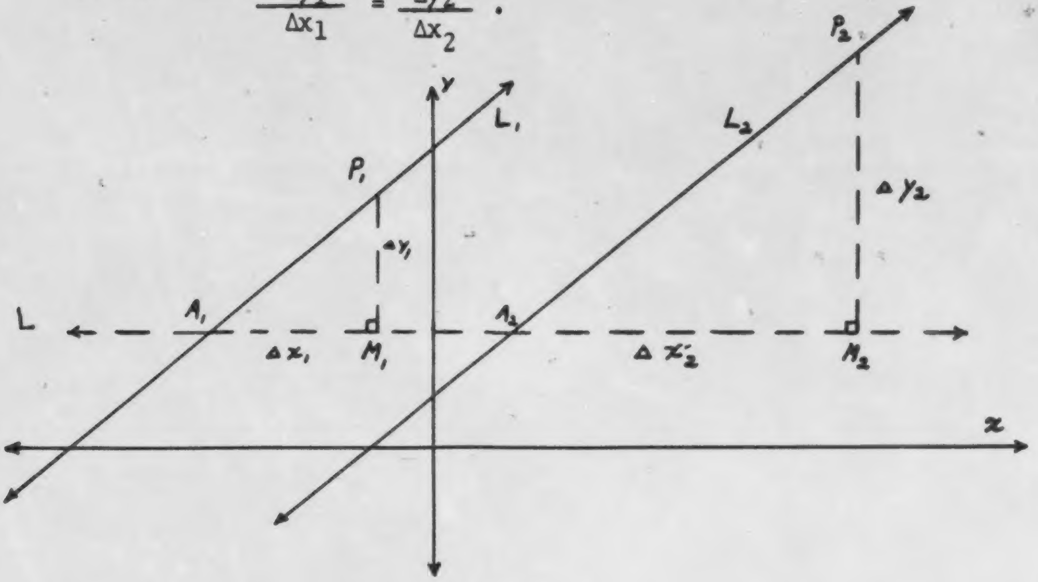


Figure 5-7-2

Thus we know that triangle $A_1P_1M_1$, and triangle $A_2P_2M_2$ are similar by the S.A.S. similarity theorem. Therefore, $\angle P_1A_1M_1$ has the same measure as $\angle P_2A_2M_2$. Thus the two lines L_1 and L_2 are cut by the transversal L in such a way that a pair of corresponding angles have the same measure. Therefore, the lines are parallel.

Part II.

If two non-vertical lines are parallel then they have the same slope. This proof is left for the student. (Problem Set 5-7-7)

It is not as obvious that if two lines are perpendicular, the product of their slopes is -1, and conversely, if lines have slopes whose product is -1, then the lines are perpendicular. Since vertical lines do not have slopes, we must exclude the case involving lines parallel to the coordinate axes.

Theorem 5-7-3

Two lines, neither of which is vertical, are perpendicular if and only if the product of their slopes equals -1.

Part I

If two non-vertical lines are perpendicular then the product of their slopes is -1.

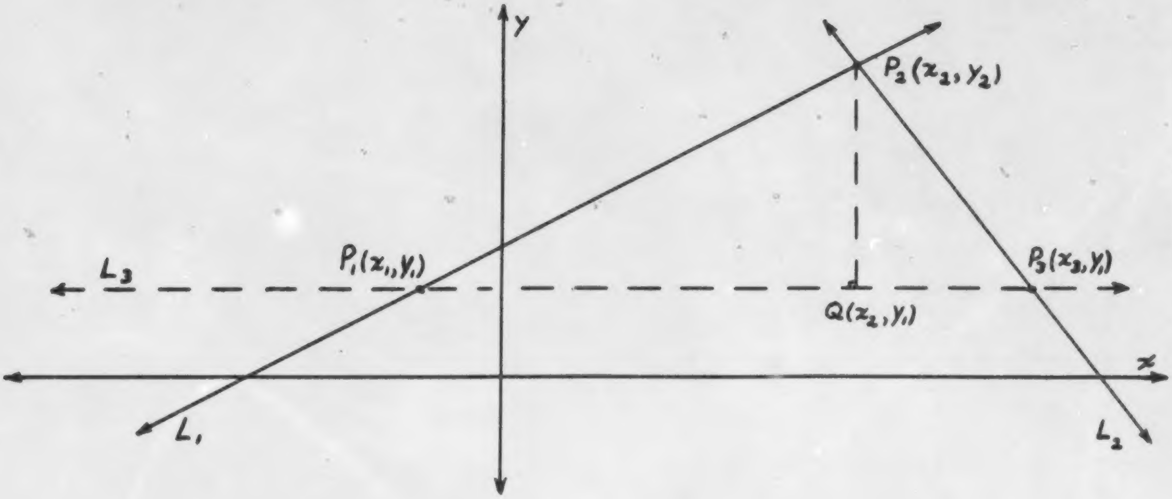


Figure 5-7-4

L_1 and L_2 are perpendicular to each other at $P_2(x_2,y_2)$.

We select a point $P_1(x_1,y_1)$ on L_1 such that $y_1<y_2$. Through this point P_1 on L_1, draw a horizontal line L_3. L_3 will intersect L_2 in some point $P_3(x_3,y_1)$. From P_2 draw a perpendicular to L_3, call this point of intersection Q. Then Q must be between P_1 and P_3.

In order to prove that Q lies between P_1 and P_3 assume that it does not. That is assume first that P_3 lies between P_1 and Q.

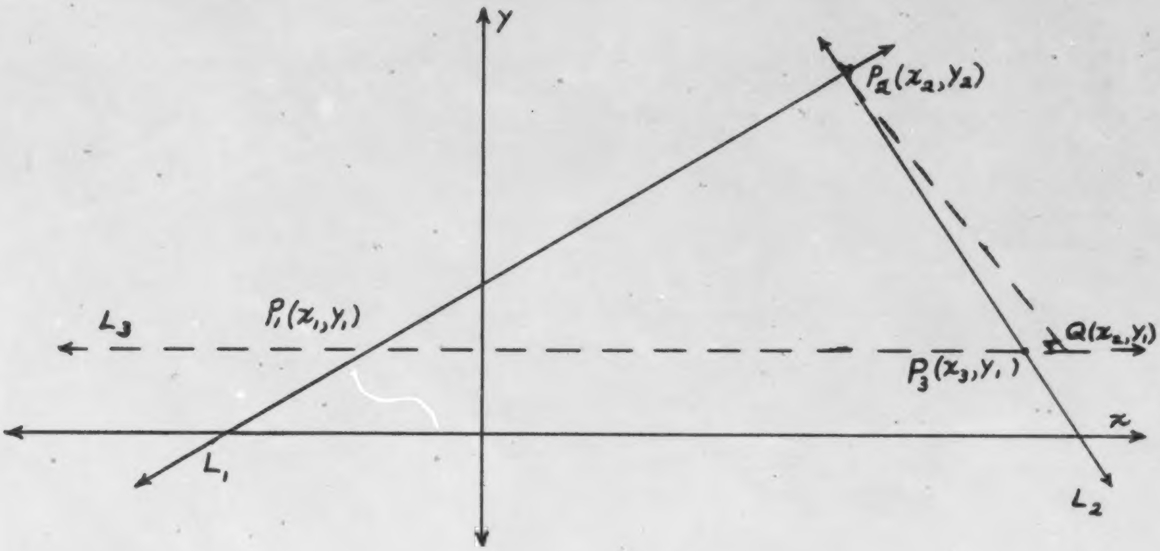


Figure 5-7-5

Line P_2Q is perpendicular to L_3. Therefore $\angle P_2QP_3$ is a right angle. However, in $\Delta P_1P_2P_3$, $\angle P_1P_2P_3$ is a right angle. This implies that $\angle P_2P_3P_1$ must be acute. A contradiction is reached, namely that $\angle P_2P_3P_1$ (an acute angle) is greater than $\angle P_2QP_3$ (right angle), since an exterior angle of a triangle is greater than either remote interior angle. If the assumption is made that Q lies on L_3 such that P_1 is between Q and P_3 an analogous contradiction is reached, i.e. $\angle P_2P_1P_3 > P_2QP_1$. Therefore Q must lie between P_1 and P_3.

Look again at Figure 5-7-4. Since the altitude to the hypotenuse of a right triangle forms two similar triangles, $\Delta P_1P_2Q \sim \Delta P_2P_3Q$. Since corresponding sides of similar triangles are proportional it follows that:

$$\frac{P_1Q}{QP_2} = \frac{QP_2}{QP_3}$$

Substituting the distances for the indicated measures of the line segments, we arrive at the following proportion:

$$\frac{|x_2 - x_1|}{|y_2 - y_1|} = \frac{|y_2 - y_1|}{|x_3 - x_2|}$$

Notice that all of these are positive numbers, that is $x_2 - x_1 > 0$, $y_2 - y_1 > 0$, $x_3 - x_2 > 0$. The absolute value signs can then be eliminated.

$$\frac{x_2 - x_1}{y_2 - y_1} = \frac{y_2 - y_1}{x_3 - x_2}$$

$$\frac{x_2 - x_1}{y_2 - y_1} = \frac{y_2 - y_1}{-(x_2 - x_3)}$$

$$\frac{1}{\frac{y_2 - y_1}{x_2 - x_1}} = -\frac{y_2 - y_1}{x_2 - x_3}$$

Since the slope, m, of $L_1 = \frac{y_2 - y_1}{x_2 - x_1}$ and the slope, m_2, of L_2 is $\frac{y_2 - y_1}{x_2 - x_3}$.

the proportion becomes

$$\frac{1}{m_1} = -m_2 \text{ or } m_1 m_2 = -1$$

We have now proved "The product of the slopes of non-vertical perpendicular lines is -1."

The proof of the converse "If the product of the slopes of two non-vertical lines is -1 then the lines are perpendicular." is left as problem 5 of Problem Set 5-7-7

Exercise 5-7-6

1.* Write a computer program which will print the general form of the equation of the line which passes through any given point in the Cartesian plane and is parallel to a given line.

2. Apply the above program to the following equation-point combinations:

   a. Parallel to $4x - y = 6$ and containing (0,0).

   b. Parallel to $y = \frac{3}{4}x + 5$, containing (2/3,0).

   c. Parallel to $2x + 7y - 10 = 0$, containing (-1,3).

   d. Parallel to $x = 7$, containing (7,2).

   e. Parallel to $y + \frac{5}{2} = 0$, containing (-3,-5).

   f. Parallel to $\frac{1}{3}x + y = 0$, containing (6,6).

3* Write a computer program which will print the general form of the equation of the line which passes through any given point on the Cartesian plane and is perpendicular to a given line.

4. Apply the program in problem 3 to the following equation-point combinations:

   a. Perpendicular to $2x - y = 6$, containing (-1,3).

   b. Perpendicular to $\frac{2}{3}x - y = 0$, containing (6,-6).

   c. Perpendicular to $x - \frac{3}{4} = 0$, containing (3,9).

   d. Perpendicular to $y = \frac{4}{3}x + 5$, containing (0,0).

   e. Perpendicular to $2x + 3y = -5$, containing (6,2).

   f. Perpendicular to $y = 7.38$, containing $(-3, -\frac{5}{2})$.

* You may want to incorporate problem 1 and 3 into the same problem, printing the equations of two lines, one parallel and one perpendicular, to the given line through the point given.

Problem Set 5-7-7

NOTE: The following problems make interesting computer applications.

1. Show that the triangles with the following vertices are right triangles:

   a. A(-4,-2), B(2,-8), C(4,6)

   b. D(2,3), E(5,3), F(5,7)

2. The vertices of a quadrilateral are A(0,0), B(1,2), C(6,5), and D(5,3). Show that ABCD is a parallelogram.

3. The vertices of a quadrilateral are A(-1,3), B(-2,-1), C(2,-2), and D(3,2). Show that:

   a. $\overline{AD}||\overline{BC}$

   b. $\overline{AB}||\overline{DC}$

   c. $\overline{AD}\perp\overline{DC}$

   d. $\overline{AC}\perp\overline{BD}$

   e. Classify the quadrilateral as specifically as possible.

4. Write equations of the lines containing the altitudes of the triangles whose vertices are:

   a. A(-6,-8), B(6,4), C(-6,10)

   b. D(-5,-5), E(5,-5), F(3,3)

   c. G(-5,2), H(9,-2), I(5,6)

5. Prove Part II of Theorem 5-7-3.

5-8 Systems of Equations.

Remember, that in previous sections, the problem of finding the center of the circle containing three given points was proposed. So far, we have studied the method of finding:

1. the slopes of the line segments joining pairs of points,

2. the coordinates of the midpoints of line segments and

3. the equations of the perpendicular-bisectors of segments.

You should recall from your study of geometry that the intersection of the perpendicular bisectors of the sides of a triangle will determine the center of the circumscribed circle. Once the coordinates of the center are found then the problem of determining the length of the radius arises. This last problem will be treated in a later section. Let's turn our attention to finding the center of the circle.

If the center of the circle exists, it will be the intersection of the perpendicular bisectors discussed above. Since two lines determine a point it will be necessary to find the point of intersection of two of these perpendicular bisectors. Hence, our problem can be stated: Given the equations of two lines, find their point of intersection, if it exists.

We shall refer to a set of two or more equations as a system of equations. Each of the individual equations is called a component of the system. Each component may contain more than one variable. The solution set of a system of equations in two variables, x and y, is the set of all ordered pairs (x,y) in the intersection of the solution sets of the component equations. This solution set may consist of a number of elements, no elements, or exactly one element. The last situation is the one of primary concern.

The mathematical notation:

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

which means $Ax + By + C = 0$ and $Dx + Ey + F = 0$, is called a system of equations. Notice that each of these equations is a component of the system.

Suppose we are given the system of equations $3x + 2y - 13 = 0$ and $2x - 3y - 13, = 0$. There are many ordered pairs of real numbers satisfying the first equation and also many pairs satisfying the second equation. Here are two samplings of the solution sets:

$$A = \{(-9,20), (-7,17), (-2,9\tfrac{1}{2}), (1,5), (5,-1), (9,-7), \ldots\}$$

$$B = \{(-9,-10\tfrac{1}{3}), (-7,-9), (-2,-5\tfrac{2}{3}), (1, -3\tfrac{2}{3}), (5,-1), (9,1\tfrac{2}{3}), \ldots\}$$

Notice that both sets have the element (5,-1) in common. This is an element in the solution set of the system $\begin{cases} 3x + 2y - 13 = 0 \\ 2x - 3y - 13. = 0 \end{cases}$ The point (5,-1) is, in fact, the only element in this solution set. These lines are obviously not the same line since their slopes are different. Therefore, they can have only this one point in their intersection.

Let us introduce a new notation for linear equations: $f(x,y) = 0$ where $f(x,y)$ represents a general expression $Ax + By + C$. Hence $f(x,y) = 0$ represents some equation $Ax + By + C = 0$.

Definition 5-8-1 Solution Set of an Equation

The Solution Set of the equation $f(x,y) = 0$ is

$$\{(a,b) \mid f(a,b) = 0\}$$

Definition 5-8-2 System of Equations

The equations $f(x,y) = 0$ and $g(x,y) = 0$ compose a second order system of linear equations, written

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$$

In Definition 5-8-2 $f(x,y) = 0$ represents some equation $Ax + By + C = 0$ and $g(x,y) = 0$ represents some other equation $Dx + Ey + F = 0$.

Definition 5-8-3 Solution Set of a System of Equations.

The solution set of the system $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ is

$$\{(a,b)\,|\,f(a,b) = 0\} \cap \{(c,d)\,|\,g(c,d) = 0\}$$

Definition 5-8-4 Linear Combination of Two Equations

Given the system of equations: $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ a linear combination

of these equations is

$$a(f(x,y)) + b(g(x,y)) = 0, \ a \neq 0 \text{ and } b \neq 0.$$

Example:

$$\begin{cases} 3x + 2y - 13 = 0 \\ 2x + 3y + 7 = 0 \end{cases}$$

Some linear combinations are:

a. $2(3x + 2y - 13) + 7(2x + 3y + 7) = 0$

or $20x + 21y + 23 = 0$

b. $-3(3x + 2y - 13) + -5(2x + 3y + 7) = 0$

or $-19x - 21y + 4 = 0$

c. $-1(3x + 2y - 13) + 2(2x + 3y + 7) = 0$

or $x + 4y + 27 = 0$

d. $5(3x + 2y - 13) + -4(2x + 3y + 7) = 0$

or $7x - 2y - 93 = 0$

Definition 5-8-5 Equivalent Systems.

Two systems of equations are said to be equivalent if and only if they have the same solution set.

Exercise 5-8-6

1. If the components of a system of equations are $3x - y + 2 = 0$ and $2x + y - 12 = 0$ find the linear combination of these components given the following values of a and b.

a	b
1	1
-3	5
4	-7
-3	-2
4	4
8.25	-7.125

2. Write a program which will print out the resultant linear combination of the equations in Problem 1 using the values of a and b from Problem 1 and any others you select.

3. Graph, on the same coordinate axes, the equations of the original system and those resultant linear combinations found in Problem 2

4. What observation can we make concerning the relative values of a and b which produce a linear combination having a slope very close to zero? A slope which is very large?

5 -9 Linear Combinations.

Certainly one generalization we can draw from Exercise 5-8-6 is the fact that all linear combinations of the system of equations seemed to pass through the point whose coordinates satisfy the system. Let us now state this conclusion as a theorem and prove it.

Theorem 5-9-1

If (r,s) satisfies the system of equations $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$

then (r,s) is an element of the solution set of any linear combination:

$$a(f(x,y)) + b(g(x,y)) = 0, \ a \neq 0 \text{ and } b \neq 0.$$

Proof:

If (r,s) satisfies the system $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$

then $f(r,s) = 0$ and $g(r,s) = 0$.

It now follows by MPE and AId that

$$a(f(r,s)) = 0 \text{ and } b(g(r,s)) = 0, \ a \neq 0 \text{ and } b \neq 0.$$

and therefore by APE and AId we conclude:

$$a(f(r,s)) + b(g(r,s)) = 0.$$

This verifies that (r,s) is an element of the solution set of the resultant linear combination.

Two more principles must be understood before we can logically generate the ordered pair which satisfies a system of equations.

Theorem 5-9-2 Principle of Linear Combination

The system of equations $f(x,y) = 0$ and $g(x,y) = 0$ is equivalent to the system obtained by pairing either of the original component equations with a linear combination of these components.

Theorem 5-9-3 Principle of Equivalent Equations

If either of the component equations of a system $f(x,y) = 0$ and $g(x,y) = 0$ is replaced by an equivalent equation, the resulting system is equivalent to the original system.

Proof of these theorems is left to those students interested.

Using the definitions and theorems we have up to now, let's look at a formal method for finding the solution set of a second order system of equations. This will be outlined for both a general system as well as a specific system. In each example, the successive steps represent the replacement of a system of equations with an equivalent system by applications of Theorems 5-9-2 and 5-9-3.

Step	Example 1	Step	Example 2
1.	$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$	1.	$\begin{cases} 3x - y - 8 = 0 \\ x + 2y - 5 = 0 \end{cases}$
2.	$\begin{cases} Ax + By + C = 0 \\ E(Ax + By + C) + (-B)(Dx + Ey + F) = 0 \end{cases}$	2.	$\begin{cases} 3x - y - 8 = 0 \\ 2(3x - y - 8) + 1(x + 2y - 5) = 0 \end{cases}$
3.	$\begin{cases} Ax + By + C = 0 \\ AEx + BEy + CE - BDx - BEy - BF = 0 \end{cases}$	3.	$\begin{cases} 3x - y - 8 = 0 \\ 6x - 2y - 16 + x + 2y - 5 = 0 \end{cases}$
4.	$\begin{cases} Ax + By + C = 0 \\ AEx - BDx + BEy - BEy + CE - BF = 0 \end{cases}$	4.	$\begin{cases} 3x - y - 8 = 0 \\ 6x + x - 2y + 2y - 16 - 5 = 0 \end{cases}$
5.	$\begin{cases} Ax + By + C = 0 \\ (AE - BD)x + CE - BF = 0 \end{cases}$	5.	$\begin{cases} 3x - y - 8 = 0 \\ (6 + 1)x - 21 = 0 \end{cases}$
6.	$\begin{cases} Ax + By + C = 0 \\ x + \frac{CE - BF}{AE - BD} = 0 \end{cases}$	6.	$\begin{cases} 3x - y - 8 = 0 \\ x + \frac{-21}{7} = 0 \end{cases}$
7.	$\begin{cases} (1)(Ax + By + C) + (-A)(x + \frac{CE - DF}{AE - BD}) = 0 \\ x + \frac{CE - BF}{AE - BD} = 0 \end{cases}$	7.	$\begin{cases} (1)(3x - y - 8) + (-3)(x - 3) = 0 \\ x - 3 = 0 \end{cases}$
8.	$\begin{cases} By + C - \frac{A(CE - BF)}{AE - BD} = 0 \\ x + \frac{CE - BF}{AE - BD} = 0 \end{cases}$	8.	$\begin{cases} -3x + 3x - y + 9 - 8 = 0 \\ x - 3 = 0 \end{cases}$
9.	$\begin{cases} y + \frac{C}{B} - \frac{A(CE - BF)}{B(AE - BD)} = 0 \\ x + \frac{CE - BF}{AE - BD} = 0 \end{cases}$	9.	$\begin{cases} -y + 9 - 8 = 0 \\ x - 3 = 0 \end{cases}$

10. $\begin{cases} y + \dfrac{AF - CD}{AE - BD} = 0 \\ x + \dfrac{CE - BF}{AE - BD} = 0 \end{cases}$

$\therefore \left(\dfrac{BF - CE}{AE - BD}, \dfrac{CD - AF}{AE - BD}\right)$

is the solution set

of $\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$

10. $\begin{cases} y - 1 = 0 \\ x - 3 = 0 \end{cases}$

$\therefore (3,1)$

is the solution set

of $\begin{cases} 3x - y - 8 = 0 \\ x + 2y - 5 = 0 \end{cases}$

Exercise 5-9-4

1. Justify the steps shown in the previous examples using Theorems 5-9-2 and 5-9-3.

2. Prove that $\left(\dfrac{BF - CE}{AE - BD}, \dfrac{CD - AF}{AE - BD}\right)$ satisfies the system $\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0. \end{cases}$

3. Solve each of the following systems using Theorems 5-9-2 and 5-9-3.

a. $\begin{cases} 2x - y = -3 \\ 9 = x + y \end{cases}$

b. $\begin{cases} y = -1/4x + 2 \\ x + 4y + 2 = 0 \end{cases}$

c. $\begin{cases} y = 1/3x + 5 \\ y = 1/3x - 5 \end{cases}$

d. $\begin{cases} 2x + 2y = 100 \\ \dfrac{x}{2} + \dfrac{3y}{5} = 14 \end{cases}$

e. $\begin{cases} 2x - y = 6 \\ 4x - 2y = 5 \end{cases}$

f. $\begin{cases} \dfrac{3x + 1}{5} = \dfrac{3y + 2}{4} \\ \dfrac{2x - 1}{5} + \dfrac{3y - 2}{4} = 2 \end{cases}$

g. $\begin{cases} 3x - 2y = 1 \\ 6x - 4y = 2 \end{cases}$

h. $\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$

i. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{5} = 1 \\ 10x + 6y = 5 \end{cases}$

j. $\begin{cases} \dfrac{x - 5}{4} = \dfrac{6y}{8} \\ 2x - 3y = 5 \end{cases}$

k. $\begin{cases} \dfrac{x - y}{2} - \dfrac{x - 4y}{6} = 4 \\ \dfrac{x + y}{9} - \dfrac{x - 2y}{6} = \dfrac{22}{9} \end{cases}$

l. $\begin{cases} \dfrac{x}{a} - \dfrac{x}{b} = \dfrac{c}{ab} \\ \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{d}{ab} \end{cases}$

m. $\begin{cases} x + ay = b \\ 2x - by = a \end{cases}$

4. Write a computer program for solving any second order system of equations. Use this program to solve the following systems of equations:

a. $\begin{cases} 5x + 4y + 7 = 0 \\ 2x - 7y + 5 = 0 \end{cases}$

b. $\begin{cases} x + 3y + 9 = 0 \\ x - 3y = -3 \end{cases}$

c. $\begin{cases} 3x + 3y + 1 = 0 \\ 2x + 2y + 1 = 0 \end{cases}$

d. $\begin{cases} 4x + y = 5 \\ 2x - 3y = 13 \end{cases}$

e. $\begin{cases} 3x = 1 - 2y \\ \frac{9}{2}x - 6y = 3 \end{cases}$

f. $\begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$

g $\begin{cases} .2x - .5y = .1 \\ .4x = y - .2 \end{cases}$

h. $\begin{cases} y = \frac{2}{5}x - 1 \\ y = \frac{9}{5}x + 6 \end{cases}$

i. $\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$

j. $\begin{cases} 4x + y = 2 \\ 2x - 3y = 8 \end{cases}$

k. $\begin{cases} 3x = -3y - 4 \\ x - 6y = 3\frac{1}{3} \end{cases}$

l. $\begin{cases} 3x + 4y = 16 \\ 5x + 3y = 12 \end{cases}$

m. $\begin{cases} \frac{x}{3} + \frac{y}{6} = \frac{2}{3} \\ \frac{2x}{5} + \frac{y}{4} = \frac{1}{5} \end{cases}$

Problem Set 5-9-5

1. A cash box contains 8 more five-dollar bills than one-dollar bills. The total value of these bills is \$304. How many bills of each denomination is there?

2. A piggy bank contains 150 coins which are a mixture of dimes and nickels. If the total amount of money is \$8.50 how many dimes are there in the jar? How many nickels?

3. Find two numbers such that one is 8.6 larger than the other. Three per-cent of the larger plus four per-cent of the smaller is 1.581.

4. One sum of money is invested at 5% and a second sum at 6%. The total yearly income from both investments is \$53.75. If the rates had been reversed on the two accounts the annual income would have been increased by \$2.50. How many dollars were invested at 5%.

5. The sum of the digits of two-digit integer is 13. The number formed by reversing the digits is 9 greater than the original number. What is the remainder when the tens digit is divided by the units digit?

6. Twice the units digit of a given two-digit number is 1 more than the tens digit. If the digits are reversed, the number is decreased by 27. Find the given number.

7. A man can row downstream 6 miles in 1 hour and return in 2 hours. Find his rate in still water and the rate of the river.

8. Two men can row at the same rate in still water. They leave a town at the same time, one going up stream and the other going downstream. The one rowing upstream rows for two hours and arrives at a town that is 3 miles from the starting point. The one rowing downstream rows for 3 hours and arrives at a town that is 13.5 miles from the starting point. Find the rate of rowing in still water and the rate of the current.

9. How many pounds of nuts worth 65 cents a pound and how many pounds worth 50 cents a pound should be mixed to make 120 pounds to sell at 56 cents a pound?

10. How many pounds of milk containing 3¼% butterfat and how many pounds of cream containing 30% butterfat should be mixed to make 862.5 pounds of milk containing 4% butterfat?

11. A carpenter can build a shed in 6 hours, but his apprentice needs 16 hours to do the same job. When they work together to build the shed, the apprentice works 5 hours more than the carpenter. How long does each work?

12. One high-speed computer system can prepare the monthly payroll of a large company in 16 hours. A faster system can do the job in 12 hours. The company is considering using both computers simultaneously. If the slower computer then works 3 hours longer than the faster one, how many hours does each work when their efforts are combined? If charges on the slower computers are $2000 per hour and on the faster one, $3000 per hour which of the three ways would be the cheapest for the company?

13. What are the times of the day when the hour and minute hands of the clock coincide.

14. Find the solution set of each of the following systems of equations.

a. $\begin{cases} 1/x + 1/y = 10 \\ 1/x - 1/y = 3 \end{cases}$

b. $\begin{cases} 6x - \frac{13}{y} = 2 \\ 5x - \frac{12}{y} = 4 \end{cases}$

c. $\begin{cases} 3x^2 - 5y^2 = 4 \\ y - 4 = 0 \end{cases}$

d. $\begin{cases} \frac{1}{2}x + \frac{2}{3}y = 18 \\ \frac{3}{4}x + \frac{4}{5}y = 21 \end{cases}$

e. $\begin{cases} \frac{3}{x} - \frac{5}{y} = 6 \\ \frac{4}{x} + \frac{3}{y} = 37 \end{cases}$

f. $\begin{cases} 3x - \sqrt{y} = 9 \\ 2x + \sqrt{y} = 11 \end{cases}$

g. $\begin{cases} 3x + 2y = 27 \\ x - y = 4 \end{cases}$

h. $\begin{cases} \frac{x+y}{2} - \frac{x-y}{3} = 8 \\ \frac{x+y}{3} + \frac{x-y}{4} = 11 \end{cases}$

15. Find an equation of the line which passes through the point (5,4) and the intersection of the lines whose equations are $y = -\frac{1}{4}x + \frac{1}{2}$ and $x + \frac{3}{2}y = -\frac{1}{2}$

16. Prove that if $a_1b_2 - a_2b_1 = 0, b_1c_2 - b_2c_1 = 0$, and $a_2c_1 - a_1c_2 = 0$, then there exists a real number $k \neq 0$ such that $a_1 = ka_2$, $b_1 = kb_2$, and $c_1 = kc_2$. Assume, of course, that $a_1b_1 \neq 0$ and $a_2b_2 \neq 0$.

5-10 The Distance Formula.

With respect to Great Goody, introduced in Section 5-5, we are now faced with finding the radius of the circle, that is the distance from one of the three non-collinear points to the center point. When the coordinates of two-points are known, the distance between them can always be found. We shall develop a formula for this purpose. Distances, as considered here, are positive.

Let us suppose that the coordinates of any two different points are known to be (x_1,y_1) and (x_2,y_2). If the first coordinates happen to be the same, that is, if $x_1 = x_2$, then the points are on a vertical line such as in Figure 5-10-2.

Definition 5-10-1

The distance between points (x_1,y_1) and (x_2,y_2) is $|y_1 - y_2|$ if and only if $x_1 = x_2$.

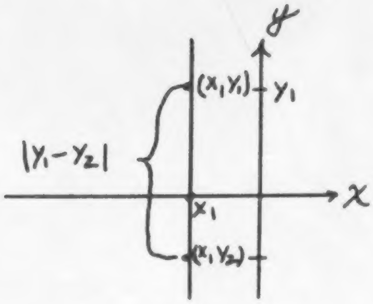


Figure 5-10-2

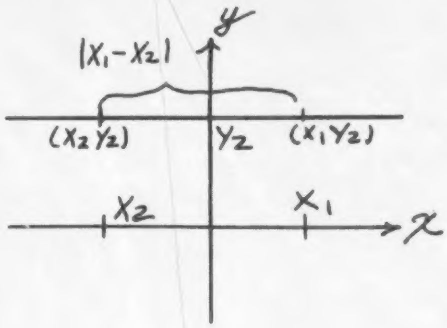


Figure 5-10-3

In a similar way, we see that if the second coordinates of two points are the same, the points lie on a horizontal line such as in Figure 5-10-3.

Definition 5-10-4

The distance between the points (x_1,y_1) and (x_2,y_2) is $|x_1 - x_2|$ if only if $y_1 = y_2$.

If the points are not on a horizontal or a vertical line, then consider the right triangle, as shown in Figure 5-10-5. The legs of the triangle have lengths $|x_1 - x_2|$ and $|y_1 - y_2|$, respectively. The distance between the points (x_1,y_1) and (x_2,y_2) is then the length of the hypotenuse, which we may find by the theorem of Pythagoras as follows:

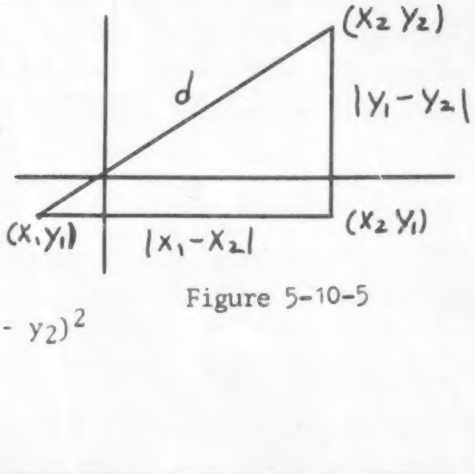


Figure 5-10-5

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

In this equation, the absolute value symbols are not necessary since second powers of real numbers are never negative. It should also be noted that the differences of the coordinates may be taken in either order, since for any real number a, $a^2 = (-a)^2$.

From the equation above we may now define distance as follows:

Definition 5-10-6 Distance Formula

The distance between any two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Notice that even if the first or second coordinates are the same, the formula still holds. Therefore the distance formula is completely general, and provides a method of finding the distance between any two points on a plane.

Example 5-10-7

Find the distance between the points (-1,-1) and (-4,3).

$$\begin{aligned} d &= \sqrt{(-1 - (-4))^2 + (-1 - 3)^2} \\ &= \sqrt{9 + 16} \\ &= 5 \end{aligned}$$

Example 5-10-8

Show that the points (2,2), (6,6), and (11,1) are the vertices of a right triangle.

We find the square of the distance between each pair of points:

$$d_1^2 = (6 - 2)^2 + (6 - 2)^2 = 32$$

$$d_2^2 = (11 - 2)^2 + (1 - 2)^2 = 82$$

$$d_3^2 = (11 - 6)^2 + (1 - 6)^2 = 50$$

Since $d_3^2 + d_1^2 = d_2^2$, we know, by the converse of the Pythagorean theorem, that the triangle is a right triangle.

The preceding example could also have been solved by showing that the product of the slopes of two of the sides of the triangle is -1.

Exercise 5-10-9

1. Write a computer program which will find the distance between any two points on the Cartesian plane.

2. Using the above program, find the distance between each of the following pairs of points:

   a. (-2,-3), (1,1)

   b. (-5,3), (7,-2)

   c. (0,4.5), (0,3.75)

   d. (-2,-2), (2,2)

   e. (0,-3), (-8.73,-3)

   f. (-98.371,24.7), (-99,25.973)

   g. (-3489,-274), (8976,4279)

   h. (a,-3), (2a,5)

3. Solve, by hand, for the distance between the points in Exercise 2a - 2d and 2h. The answer you obtain should be in simplest radical form.

4. Write a program which will compute the perimeter of any pclygon given the coordinates of its vertices.

5. Use the program written in Problem 4 to find the following perimeters:

   a. Triangle ABC, given A(-2,3), B(1,4), and C(-4,9)

   b. Quadrilateral MNPQ, given M(-3,-1), N(5,3), P(4,-1), and Q(0,-3)

   c. Octagon given the vertices (-7,-1), (3,-7), (-5,-4), (8,-1), (9,2), (3,3), (-1,9), (-5,3).

6. Write a computer program to test whether the following are scalene, isosceles, equilateral, or right triangles. Be certain to observe that some triangles may satisfy more than one of these conditions, i.e., isosoceles right triangle.

   a. A(15,4), B(-7,8), and C(-1,-4)

   b. D(1,4), F(2,1), and F(3,2)

   c. G(0,0), H(10,-4), and I(2,5)

   d. J(0,0), K(9,-2), and L(2,4)

   e. M(2,1), N(7,4), and P(2,7)

Problem 5-10-10

1. Using the above program, find the distance between each of the following pairs of points:
   - a. (-2,-3), (1,1)
   - b. (-5,3), (7,-2)
   - c. (0,4.5), (0,3.75)
   - d. (-2,-2), (2,2)
   - e. (0,-3), (-8.73,-3)
   - f. (-98.371,24.7), (-99,25.973)
   - g. (-3489,-274), (8976,4279)
   - h. (a,-3), (2a,5)
2. The vertices of a quadrilateral are I(-2,1), J(3,3), K(3,6), and L(-2,4).
   - a. Show that IJKL is a parallelogram by showing that $\overline{IJ} \parallel \overline{LK}$ and that $\overline{IJ}$ and $\overline{LK}$ have the same length.
   - b. Determine whether the diagonals are perpendicular and whether they have the same length.
3. The vertices of a quadrilateral are C(-4,-1), D(2,-6), F(7,0), and J(1,5).
   - a. Show that CDFJ is equilateral.
   - b. Show that its diagonals are perpendicular and have the same length.
4. Given the coordinates of the vertices of any quadrilateral, write a computer program to determine the specific type of quadrilateral the points describe. Test your program with the following data:
   - a. A(1,4), B(3,2), C(4,5), D(2,8)
   - b. E(-8,3), F(-4.5,7.5), G(-3,2), H(-6.5,-2.5)
   - c. J(2,4), K(6,0), L(2,-4), M(-2,0)
   - d. N(1,5), P(2,4), Q(-1,1), R(-2,2)
5. Write a computer program which will compute the area of a convex polygon given the coordinates of the vertices.

5-11 A Problem Reassessed

You are now equipped to solve "Great Goody", which is restated as follows:

> Given the coordinates of any three points, determine if they are collinear or not. If not, determine the center and radius of the circle containing the given points.

Exercise 5-11-1

Using the topics discussed previously in this chapter write and execute a program to solve "Great Goody" with the following data. Be sure your program will work for <u>any</u> combinations of three given points.

```
DATA  (0,0), (4,0), (-5,0)

DATA  (-3,-2), (-3,0), (-3,6)

DATA  (6,2), (6,-1), (8,-1)

DATA  (1,1), (1,8), (6,8)

DATA  (-2,5), (1,2), (6,2)

DATA  (4,3), (6,3), (8,1)

DATA  (5,3), (9,7), (12,11)

DATA  (4,6), (10,12), (2,10)

DATA  (4,4), (4,4), (4,4)

DATA  (14,12), (35,55), (14056,22088)
```

5-12 (optional) The Distance from a Point to a Line.

Another formula for distance which is occasionally convenient is the one for calculating the distance from a point to a line.

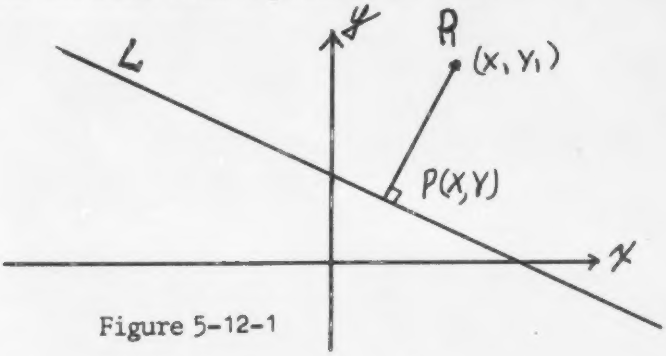


Figure 5-12-1

Given a point $P_1(x_1,y_1)$ and an oblique line L with the equation $Ax + By + C = 0$. Draw a perpendicular from P_1 to line L. Let $P(x,y)$ be the point of intersection of the perpendicular and the line.

The slope of L is $-\frac{A}{B}$. Therefore the slope of the perpendicular $\overline{PP_1}$ will be $\frac{B}{A}$. The equation of the line containing P_1 and P is

$$\frac{B}{A} = \frac{y - y_1}{x - x_1} \qquad x \neq x_1$$

Performing the appropriate transformations on the equation of this line we get:

$$Bx - Bx_1 = Ay - Ay_1 \qquad x \neq x_1$$

This equation yields the following system when it is combined with the equation of the line:

$$\begin{cases} Bx - Ay + Ay_1 - Bx_1 = 0 \\ Ax + By + C = 0 \end{cases}$$

Transforming this system we get:

$$\begin{cases} x = \dfrac{B^2x_1 - ABy_1 - AC}{A^2 + B^2} \\ y = \dfrac{-ABx_1 + A^2y_1 - BC}{A^2 + B^2} \end{cases}$$

The solution set of this system is:

$$\left\{\left(\frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}, \frac{-ABx_1 + A^2y_1 - BC}{A^2 + B^2}\right)\right\}$$

This ordered pair represents the coordinates of the intersection of the given line L with the perpendicular.

The distance from P to P_1 can now be found using the distance formula,

$$d = \sqrt{\left(\frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2} - x_1\right) + \left(\frac{-ABx_1 + A^2y_1 - BC}{A^2 + B^2} - y_1\right)^2}$$

$$d = \sqrt{\left(\frac{B^2x_1 - ABy_1 - AC - x_1(A^2 + B^2)}{A^2 + B^2}\right)^2 + \left(\frac{-ABx_1 + A^2y_1 - BC - y_1(A^2 + B^2)}{A^2 + B^2}\right)^2}$$

$$d = \sqrt{\left(\frac{B^2x_1 - ABy_1 - AC - A^2x_1 - B^2x_1}{A^2 + B^2}\right)^2 + \left(\frac{-ABx_1 + A^2y_1 - BC - A^2y_1 - B^2y_1}{A^2 + B^2}\right)^2}$$

$$d = \sqrt{\left(\frac{-A(Ax_1 + By_1 + C}{A^2 + B^2}\right)^2 + \left(\frac{-B(Ax_1 + By_1 + C)}{A^2 + B^2}\right)^2}$$

$$d = \sqrt{\frac{A^2(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2} + \frac{B^2(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2}}$$

$$d = \sqrt{\frac{(A^2 + B^2)(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2}}$$

$$d = \sqrt{\frac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}}$$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Thus we have developed a formula to find the distance from $P_1(x_1,y_1)$ to the line L having the equation $Ax + By + C = 0$.

Problem Set 5-12-2

1. Find the distance from each given point to the given line:
    a. $(4,5)$, $7x + 24y - 21 = 0$
    b. $(-3,5)$, $4x - 3y + 12 = 0$
    c. $(4,0)$, $2x - 3y - 2 = 0$
    d. $(-4,3)$, $2y - 1 = 0$
    e. $(0,0)$, $8x - 3 = 0$
2. Write a computer program to solve all parts of problem 1.
3. Find the distance between the parallel lines $3x - 4y + 12 = 0$ and $3x - 4y + 22 = 0$. Hint: Notice that $(0,3)$ is contained in one of the lines.
4. Find the distance between the parallel lines $2x - 3y = 6$ and $2x - 3y - 9 = 0$.
5. Write equations of the bisectors of the angles formed by the lines having equations $3x - 4y + 6 = 0$ and $12x - 5y - 9 = 0$. Hint: Since every point (x,y) on each bisector is equidistant from the two lines,

    then
    $$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|A'x_1 + B'y_1 + C'|}{\sqrt{(A')^2 + (B')^2}}$$

    or
    $$\frac{3x - 4y + 6}{5} = \pm\frac{12x - 5y - 9}{15}$$
6. Write equations of the bisectors of the angles formed by the lines having equations $3x + 4y + 6 = 0$ and $5x + 12y - 9 = 0$.
7. Write equations of the bisectors of the angles formed by the lines having equations $x - 3y = 6$ and $6x - y = 2$.
8. The vertices of a triangle are $A(-2,-3)$, $B(4,3)$, and $C(-3,4)$. Find an equation of the bisector of $\angle C$.

5-13 Nth Order Systems of Equations

We have recently developed the logic in finding the solution set of a system of second order linear equations.

$$\begin{cases} ax + by + c = 0 \\ dx + ey + f = 0 \end{cases}$$

This was accomplished by the application of the following definitions and principles:

(a) Principle of Equivalent Systems. Two systems of equations are equivalent if and only if they have the same solution set.

(b) Principle of Equivalent Equations. If either of the component equations of a system of equations is replaced by an equivalent equation, the resulting system is equivalent to the original system.

(c) Principle of Linear Combination. A system of equations is equivalent to the system obtained by pairing either of the original component equations with a linear combination of these components.

At the risk of being repetitive, let us take another look at the application of these principles on a general system of linear equations in standard form.

$$\begin{cases} Ax + By = C \\ Dx + Ey = F \end{cases}$$

$$\begin{cases} 1x + \frac{B}{A}y = \frac{C}{A} \qquad (A \neq 0) \\ Dx + Ey = F \end{cases}$$

$$\begin{cases} 1x + \frac{B}{A}y = \frac{C}{A} \\ (D + -D)x + (E + -\frac{BD}{A})y = F + (-\frac{DC}{A}) \end{cases}$$

$$\begin{cases} 1x + \frac{B}{A}y = \frac{C}{A} \\ 0x + \frac{AE - BD}{A}y = \frac{AF - CD}{A} \end{cases}$$

$$\begin{cases} 1x + \frac{B}{A}y = \frac{C}{A} \\ 0x + 1y = \frac{AF - CD}{AE - BD} \qquad (AE \neq BD) \end{cases}$$

$$\begin{cases} (1 + 0)x + (\frac{B}{A} + -\frac{B}{A})y = \frac{C}{A} + -\frac{B(AF - CD)}{A(AE - BD)} \\ 0x + 1y = \frac{AF - CD}{AE - BD} \end{cases}$$

$$\begin{cases} 1x + 0y = \frac{CE - BF}{AE - BD} \\ 0x + 1y = \frac{AF - CD}{AE - BD} \end{cases}$$

$\therefore (\frac{CE - BE}{AE - BD}, \frac{AF - CD}{AE - BD})$ is the solution of

$$\begin{cases} Ax + By = C \\ Dx + Ey = F \end{cases}$$

if such a solution exists, i.e., $AE \neq BD$, $A \neq 0$

Notice that, in this sequence of steps, we have consistently placed the components, linear combinations, and equivalent equations in the same standard form as the original components of the system. That is, the variable x and its coefficient are always written as the first term in the equations. The variable y and its coefficient are the second term in the equation. The constants are found to be to the right of the equal sign.

If we now agree to be consistent in positioning the terms of an equation, we can greatly simplify the work with systems. We will use an array of numbers to represent a system of equations.

$$\begin{cases} Ax + By = C \\ Dx + Ey = F \end{cases} \longleftrightarrow \begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix}$$

This example suggests an array which is an abbreviation for the system. The elements in the first column (vertical) represent the coefficients of the variable x. The second column of elements represent the coefficients of the variable y and the last column contains the constants. The individual rows (horizontal) represent the component equations.

We also have a method by which we may refer to the individual coefficient of either equation. In this case, the element, A, in the first row, first column is the coefficient of x in the first component equation. The element in the second row, third column, is the constant, F, in the second component equation. In general, we could incorporate the notation Aij to mean the element found in the ith row (horizontal) and jth column (vertical). This double subscript notation will simplify our later discussions concerning arrays of this type.

Example 5-13-1

Let the system $\begin{cases} Ax + By = C \\ Dx + Ey = F \end{cases}$

be represented by the array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

then

a_{11} represents A

a_{12} represents B

a_{13} represents C

a_{21} represents D

a_{22} represents E

and a_{23} represents F

Exercise 5-13-2

1. Write the array of numbers which will represent each of the following systems of equations:

a. $\begin{cases} 3x + 4y = 7 \\ -5x + 2y = 1 \end{cases}$

b. $\begin{cases} 2x - 3y = 8 \\ 9x + 7y = -9 \end{cases}$

c. $\begin{cases} x + 0y = 4 \\ 0x + y = -5 \end{cases}$

d. $\begin{cases} 2x = 3y - 7 \\ x + 5y + 2 = 0 \end{cases}$

2. Given the abbreviation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

for the system

$$\begin{cases} x - 2y - z = 3 \\ -5x + 4y - 7z = 2 \\ 17x + 0y + 6z = -11 \end{cases}$$

name the coefficients which correspond to the following -a_{ij}'s:

(a) a_{12}

(b) a_{21}

(c) a_{33}

(d) a_{24}

(e) a_{13}

(f) a_{11}

(g) a_{31}

(h) a_{14}

(i) a_{23}

(j) a_{34}

3. What is the system of equations which is suggested by the array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

where $a_{11} = 6$, $a_{22} = -3$, $a_{13} = 2$, $a_{21} = 5$, $a_{12} = 6$, $a_{23} = -1$

Programming Double Subscripts

In the previous exercise, Exercise 5-13-2., and the discussion preceeding it we have referred to a single element by its position in the array. This position was identified by two counting numbers.

Example 5-13-3

If $a_{23} = 7$ we associate the number 7 with the element in the second row and the third column.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 7 \end{pmatrix}$$

By using nested loops we can establish double subscript notation in a program which is analogous to our agreed array notation.

If we were working with only one row, normally we would use single subscripts to read and print a 1 x 6 array.

Program 5-13-4

```
10 DIM A(6)
20 FOR J = 1 TO 6
30 READ A(J)
40 PRINT "A(";J;") = "; A(J)
50 NEXT J
60 DATA 2, 4, -3, 7, 1, -2
70 END
```

We could take the same situation and apply double subscripts. Since we are discussing a 1 x 6 array, however, each first subscript will be 1.

Program 5-13-5

```
10 DIM A(1, 6)
20 FOR J = 1 TO 6
30 READ A(1,J)
40 PRINT "A(1, ";J;") ="; A(1, J)
50 NEXT J
60 DATA 2, 4, -3, 7, 1, -2
70 END
```

The differences in the output of these two programs may be analyzed in Example 5-13-6

Example 5-13-6

Program 5-13-4 : A(1) = 2 A(2) = 4 A(3) = -3 A(4) = 7 A(5) = 1 A(6) = -2

Program 5-13-5 : A(1,1) = 2 A(1,2)=4 A(1,3)=-3 A(1,4)=7 A(1,5)=1 A(1,6)= -2

If we desired to evaluate this same data as being representative of a 2 x 3 array of elements the following program would be used.

Program 5-13-7

```
 10 DIM A(2, 3)

 20 FOR I = 1 TO 2

 30 FOR J = 1 TO 3

 40 READ A(I, J)

 50 PRINT "A(";I;",";J;") = ";A(I,J)

 60 NEXT J

 70 PRINT

 80 NEXT I

 90 DATA 2, 4, -3, 7, 1, -2

100 END
```

Output (Program 5-13-7)

```
A(1,1) = 2          A(1,2) = 4          A(1,3) = -3

A(2,1) = 7          A(2,2) = 1          A(2,3) = -2
```

This output represents the 2 x 3 array:

$$\begin{pmatrix} 2 & 4 & -3 \\ 7 & 1 & -2 \end{pmatrix}$$

Exercise 5-13-8

1. Using double subscripts, write a computer program which will read the coefficients and constants of the system:

$$\begin{cases} x - 2y = 3 \\ -5x + 4y = -3 \end{cases}$$

and print the corresponding array.

2. Expanding the program written in Exercise 1, read the coefficients and constants of the system:

$$\begin{cases} 2x - 4y + 3z = 4 \\ -3x + 7z = -1 \\ x + y - 2z = 2 \end{cases}$$

and print the corresponding 3 x 4 array.

3. Write a computer program which will print the array CORRESPONDING to any system of n equations in n variables. Use double subscripts.

4. Solve the following systems of equations using the principles and definitions reviewed in this section:

(a) $$\begin{cases} 7x + 3y = 4 \\ x + y = 0 \end{cases}$$

(b) $$\begin{cases} x - 6y + 2z = 5 \\ 2x - 3y + z = 4 \\ 3x + 4y - z = -2 \end{cases}$$

Operations on Arrays

We shall agree that an array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1\ n+1} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2\ n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{n\ n+1} \end{pmatrix} \qquad (1)$$

represents a system of n equations in n variables. We shall also agree that all the equations represented are in standard form. Such an array will have n rows and n + 1 columns. The diagonal made up of the A_{ij} elements where i = j shall be referred to as the major diagonal of the array.

There are some transformation principles which we can apply to array (1) so that the solution set of the system it represents is not altered.

Transformation Principles:

Principle A. - Any two rows may be interchanged.

Explanation - The order of the equations in a system does not affect the solution set of that system.

Principle B. - Any row may be multiplied by a nonzero constant.

Explanation - The Multiplication Transformation Principle for Equations allows us to multiply both sides of an equation by a nonzero constant without changing the solution set.

Principle C. - Any row may be replaced by the sum of itself and a nonzero multiple of another row.

Explanation - The Principle of Linear Combination allows us to replace a component equation with a linear combination without altering the solution set of the system.

The student should notice the correspondence between these Transformation Principles and the principles given at the beginning of Section 5-13

With these Transformation Principles and an understanding of the meaning of the array we can now simplify the process by which we find the solution set of a system of equations. A step-by-step explanation may be seen in Example 5-13-9.

Example 5-13-9

Given the system

$$\begin{cases} 2x + 3y = 4 \\ 6x - 2y = -10. \end{cases}$$

Find its solution set.

1. $\begin{pmatrix} 2 & 3 & 4 \\ 6 & -2 & -10 \end{pmatrix}$ (1) The array which represents the system.

2. $\begin{pmatrix} 1 & \frac{3}{2} & 2 \\ 6 & -2 & -10 \end{pmatrix}$ (2) Replace Row 1 with a nonzero multiple of itself (Transformation Property B)

3. $\begin{pmatrix} 1 & \frac{3}{2} & 2 \\ 0 & -11 & -22 \end{pmatrix}$ (3) Replace Row 2 with the sum of itself and a nonzero multiple (-6) of Row 1 (Transformation Property C)

4. $\begin{pmatrix} 1 & \frac{3}{2} & 2 \\ 0 & 1 & 2 \end{pmatrix}$ (4) Replace Row 2 with a nonzero multiple $(-\frac{1}{11})$ of itself (Transformation Property B)

5. $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ (5) Replace Row 1 with the sum of itself and a nonzero multiple $(-\frac{3}{2})$ of Row 2 (Transformation Property C)

This final array represents the system

$$\begin{cases} 1x + 0y = -1 \\ 0x + 1y = 2 \end{cases}$$

Therefore $\{(-1,2)\}$, must be the solution set of the original system.

Exercise 5-13-10

1. Solve the following systems for their solution sets applying the transformation principles. A, B, and C, to the appropriate arrays.

a. $\begin{cases} 7x + 3y = 4 \\ x + y = 0 \end{cases}$

b. $\begin{cases} 3x + 4y = -5 \\ -2x + 7y = 1 \end{cases}$

c. $\begin{cases} 2y = 3 \\ -3x + y = 7 \end{cases}$

d. $\begin{cases} 3x - 2y + 9z = -14 \\ y - 3z = 7 \\ z = 1 \end{cases}$

e. $\begin{cases} x - 6y + 2z = 5 \\ 2x - 3y + z = 4 \\ 3x + 4y - z = -2 \end{cases}$

We can now appreciate the elegance in this symbolic representation of a system of equations. The Transformation Principles allow us to transform a complex array into an equivalent but simplified array. Careful examination of this transformation process reveals repetitive application of the Transformation Principles. Therefore, we should be able to computerize this technique. Through a series of flow charts we will now build a computer model of this algorithm for solving any system of equations.

Exercise 5-13-11

Write a flow chart which will convert the following array into a transformed array with 1's in the major diagonal positions and 0's in all other positions except column three. Be certain the system represented by this array is equivalent to the original system.

$$\begin{pmatrix} 2 & -3 & 4 \\ 5 & 2 & -7 \end{pmatrix}$$

The flow chart in Figure 5-13-12 represents a portion of the logic necessary to produce 1's along the major diagonal of the array. Study it carefully!

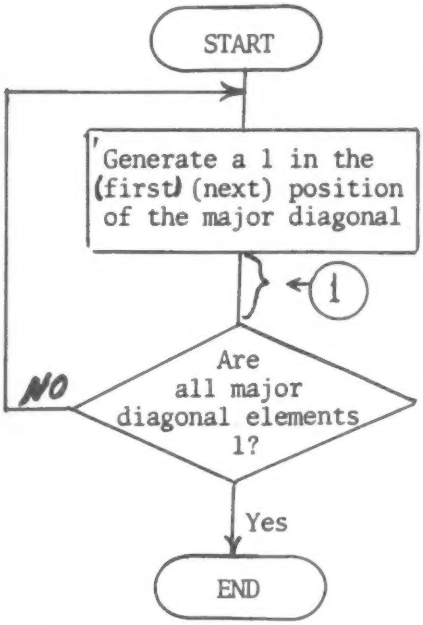

START
Generate a 1 in the (first) (next) position of the major diagonal
1
Are all major diagonal elements 1?
NO
Yes
END

Figure 5-13-12

A 1 may be generated in any diagonal position by application of Transformation Principle B. However, this principle does not permit us to multiply a single element of a row by a non-zero constant without also multiplying every other element in that row by this same constant. The missing segment, ① of the flow chart in Figure5-13-12 must be correctly written to conform with this transformation principle. That is, each element of a row must be multiplied by that factor which will generate a 1 in the diagonal position of that row.

The logic representing this process is illustrated in Figure 5-13-13

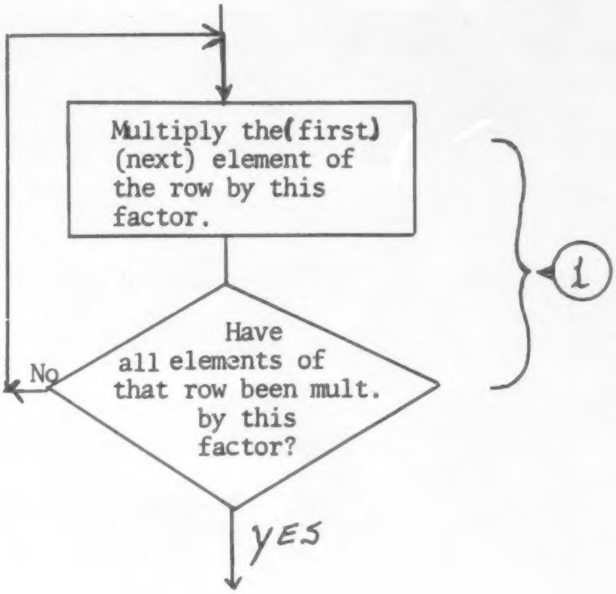

Multiply the(first) (next) element of the row by this factor.
Have all elements of that row been mult. by this factor?
No
YES
1

Figure 5-13-13

Figure 5-13-14 represents the combination of the logic in the two preceeding flow charts, figures 5-13-12 and 5-13-13.

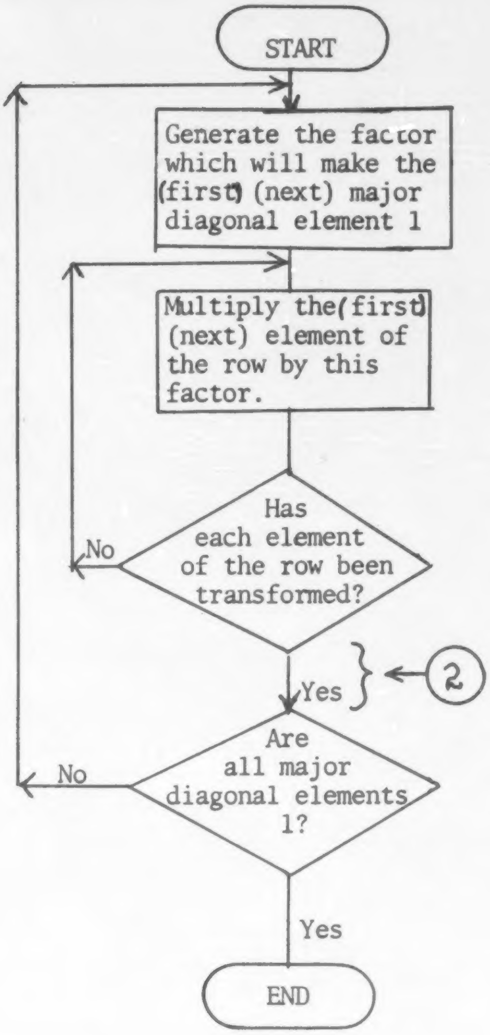

START
Generate the factor which will make the (first) (next) major diagonal element 1
Multiply the (first) (next) element of the row by this factor.
Has each element of the row been transformed?
No
Yes
2
Are all major diagonal elements 1?
No
Yes
END

Figure 5-13-14

Exercise 5-13-15

Given the following system of equations:

$$\begin{cases} -2x + 4y = 8 \\ 3x - y = 3 \end{cases}$$

write the array which represents this system. Transform it into an equivalent array with 1's along with major diagonal. Use the logic previously discussed.

Exercise 5-13-16

Using the array developed in Exercise 5-13-15

$$\begin{pmatrix} 1 & -2 & -4 \\ -3 & 1 & -3 \end{pmatrix}$$

Transform this array into an equivalent array such that $a_{21} = a_{12} = 0$ and the elements $a_{11} = a_{22} = 1$ remain unchanged throughout this transformation process.

Those of you who labored with Exercise 5-13-16 at any length, realize that such a transformation is impossible. This means that it was a waste of time in Exercise 5-13-15 to generate the 1 in position a_{22} before developing a 0 in position a_{21}.

We can now see the necessity of expanding section ② of the flow chart in Figure 5-13-14. We must chart the logic necessary to generate zeros in all the non-diagonal positions of a column before proceeding to generate a 1 in the next diagonal position. This extension of the flow chart is shown in Figure 5-13-17.

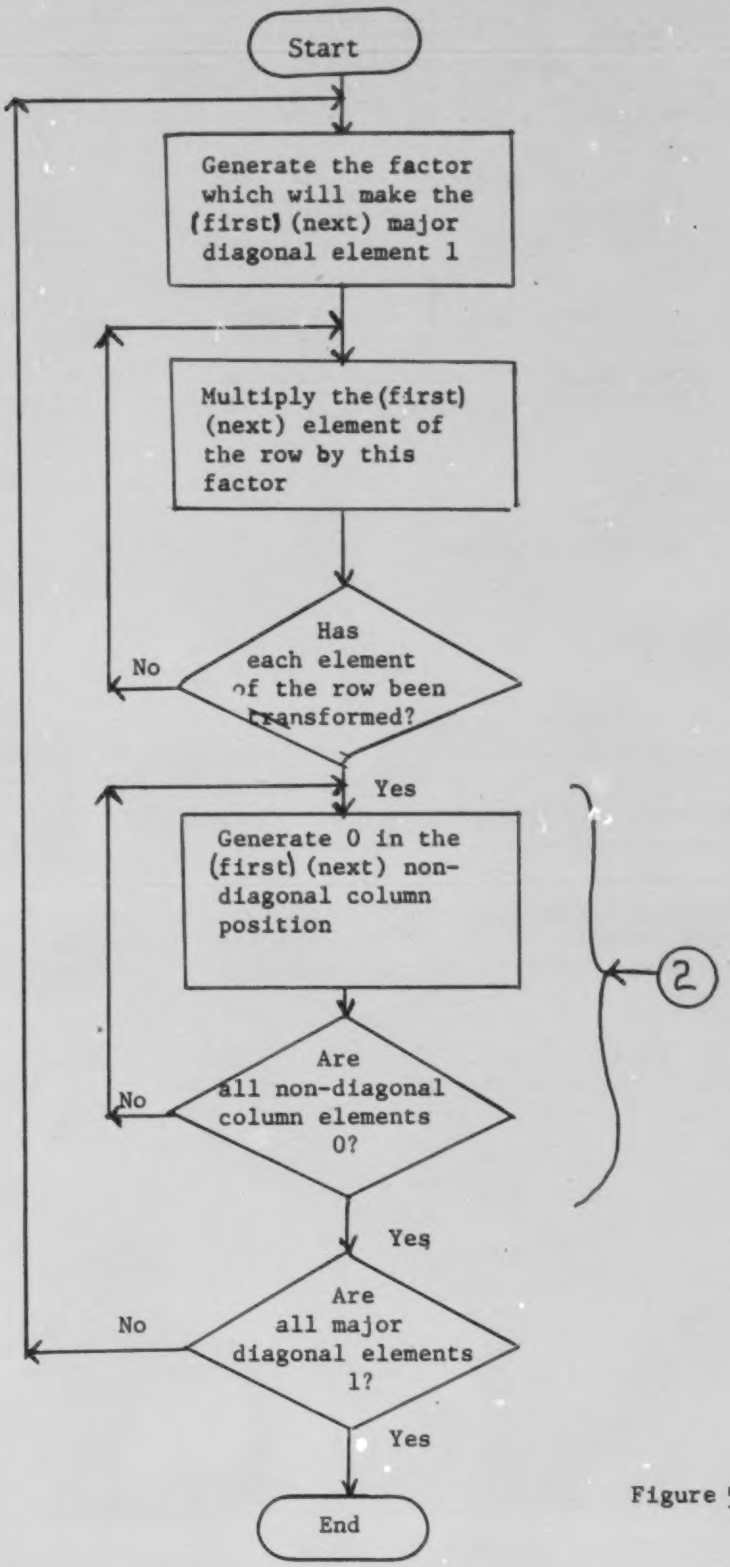

Start
Generate the factor which will make the (first) (next) major diagonal element 1
Multiply the (first) (next) element of the row by this factor
Has each element of the row been transformed?
No
Yes
Generate 0 in the (first) (next) non-diagonal column position
②
Are all non-diagonal column elements 0?
No
Yes
Are all major diagonal elements 1?
No
Yes
End

Figure 5-13-17

Exercise 5-13-18

1. Construct a computer program from the logic in this section. This program should be designed to print the solution set of the following systems of equations:

a. $$\begin{cases} 7x + 3y = 4 \\ x + y = 0 \end{cases}$$

b. $$\begin{cases} 3x + 4y = -5 \\ -2x + 7y = 1 \end{cases}$$

c. $$\begin{cases} 2y = 3 \\ -3x + y = 7 \end{cases}$$

d. $$\begin{cases} 3x - 2y + 9z = -14 \\ y - 3z = 7 \\ z = 1 \end{cases}$$

e. $$\begin{cases} x - 6y + 2z = 5 \\ 2x - 3y + z = 4 \\ 3x + 4y - z = -2 \end{cases}$$

f. $$\begin{cases} 3x - 2y = 5 \\ -6x + 4y = -10 \end{cases}$$

g. $$\begin{cases} 2x + 3y = 7 \\ 6x + 9y = -2 \end{cases}$$

2. Expand the program used in exercise one by adding the proper PRINT commands to output each equivalent array developed by the transformation properties. This will allow you to evaluate the solution set of the systems in 1f and 1g above.

3. Using this improved program find the transformed arrays and the solution sets of the following systems.

a. $$\begin{cases} 4x = -2y \\ x - y = 6 \end{cases}$$

b. $$\begin{cases} a + 2b - c = 0 \\ 2a = b - c + 5 \\ 4a + 2b = 6 - 5c \end{cases}$$

c. $$\begin{cases} 4x - 3y = 8 - z \\ x + 2y - z = 4 \\ 5x = y + 12 \end{cases}$$

d. $$\begin{cases} 2x - 3y + z = 12 \\ x = \frac{3}{2}y - \frac{1}{2}z + 6 \\ x + \frac{3}{4}y - 2z = \frac{7}{12} \end{cases}$$

e. $$\begin{cases} a + 2b + 3c + 5d = 6 \\ 2a + 3b + 4c + 8d = 2 \\ 3a + 5b + 7c + 2d = 4 \\ a + 2b + 4c + 6d = 10 \end{cases}$$

f. $$\left\{\begin{array}{rcrcrcrcrcrcrcr} a & + & 3b & + & c & + & & + & e & + & 2f & + & 4g & = & 1 \\ & & 2b & & & + & d & & & + & 2f & & & = & 3 \\ & & & & 3c & + & d & + & e & & & + & g & = & 5 \\ & & 2b & & & + & 2d & & & + & 4f & + & 3g & = & 1 \\ 2a & + & 6b & + & 2c & & & + & 6e & + & 5f & + & 9g & = & 3 \\ a & + & 5b & + & c & + & d & + & e & + & 5f & + & 6g & = & 7 \\ a & & & + & 4c & + & d & + & 2e & + & 2f & + & 8g & = & 8 \end{array}\right.$$

g. Mrs. Hawkins owns real estate in two counties. In county A the tax evaluation is \$3.12 per hundred dollars valuation. In county B the rate is \$3.28. Her total tax is \$856.24 per year and the total assessed value of her real estate is \$26,700. What is the value of her property in each county?

h. The Cains keep a coin box into which they drop money to be used for a vacation. When they counted the money they found the amount to be \$136.90, consisting of quarters, dimes and nickels. There were two times as many nickels as dimes, and 112 more quarters than dimes. How many coins of each kind were there?

5-14 Relations Defined by Compound Sentences.

Two or more simple sentences may be combined by connective words such as "and", and "or" to form new sentences called compound sentences. A compound sentence formed with the connective "and" is called a conjunction.

For example $\{x|\ -3 < x < 5\}$ is, by definition, equivalent to $\{x|\ -3 < x \text{ and } x < 5\}$. In the second set the sentence "$-3<x$ and $x<5$" is called a conjunction.

From our work with sets in Chapter one we can see that-

$$\{x|\ -3 < x \text{ and } x < 5\} = \{x|\ -3 < x\} \cap \{x|\ x < 5\}$$

The graph of these sets are shown in Figure 5-14-1

$\{x|\ -3 < x\}$

$\{x|\ x < 5\}$

$\left.\begin{array}{l}\{x|\ -3 < x\} \cap \{x|\ x < 5\} \\ \{x|\ -3 < x \text{ and } x < 5\}\end{array}\right\}$

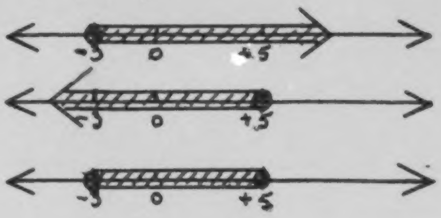


Figure 5-14-1

Definition 5-14-2

The solution set of a conjunction is the intersection of the solution sets of the individual sentences.

Example 5-14-3

We know that

$\{(3,-1)\} = \{(x,y)|x + y = 2 \text{ and } x - y = 4\} = \{(x,y)|x + y = 2\} \cap \{(x,y)\ |\ x - y = 4\}$ as shown below.

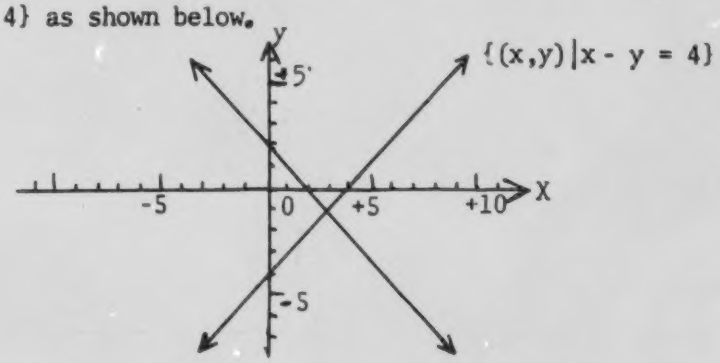
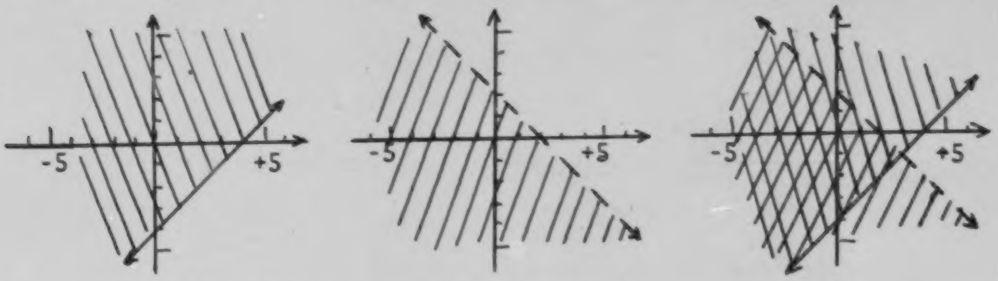


Figure 5-14-4

Example 5-14-5

Graph : $\{(x,y)|x + y < 2 \text{ and } x-y \leq 4\}$. We first graph $\{(x,y)|(x-y) \leq 4\}$. Then we graph $\{(x,y)|x + y < 2\}$, and tnen find their intersection.



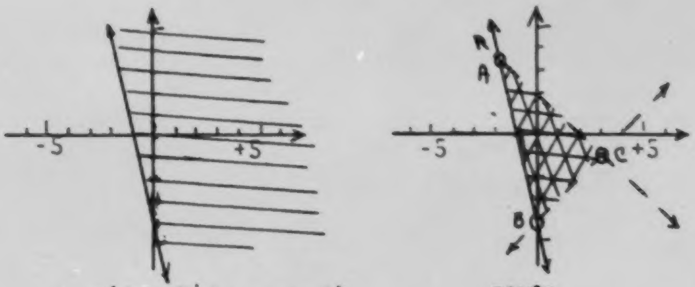
$J = \{(x,y)|x - y \leq 4\}$ $K = (x,y)|x + y < 2\}$ $J \cap K$

Figure 5-14-6

Example 5-14-7

Graph the conjunction $x + y < 2$ and $x - y \leq 4$ and $4x + y \geq -4$. That is, graph the relation $\{(x,y)|x + y < 2$ and $x - y \leq 4$ and and $4x + y \geq -4.\}$

The graph of $x + y \leq 2$ is as shown in the previous example. The graph of $x - y < 4$ is the same as that for $x - y \leq 4$, in the previous example, except that the line $x - y = 4$ is not included. The graph of $4x + y \geq -4$ is as shown at the left in Figure 5-14-8.



$L = \{(x,y)|4x + y \geq -4\}$ $J \cap K \cap L$

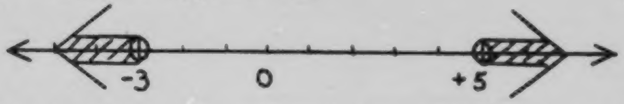
Figure 5-14-8

The graph of the conjunction is, of course, the intersection of the individual graphs. Note that this graph consists of the interior of $\triangle ABC$, together with the line segment AB except for its endpoints A and B.

A compound sentence formed by using the connective "or" is called a disjunction.

$\{x|x < -3 \text{ or } x > 5\}$ is a set whose set selector is a disjunction.

Graphically, this set appears as:



$$\{x|x < -3 \text{ or } x > 5\} = \{x|x < -3\} \cup \{x|x > 5\}$$

Figure 5-14-9

Definition 5-14-10

A disjunction of two or more sentences has as its solution set the union of the solution sets of the individual components.

Example 5-14-11

Draw a graph of $\{(x,y)|x + y < 2 \text{ or } x - y < 4\}$. The graph of this relation is the union of the graphs of the two individual relations, as shown.

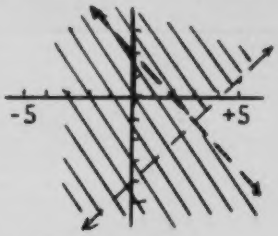


Figure 5-14-12

The graph of a relation defined by a negation of a sentence is the complement of the one defined by the sentence itself.

Example 5-14-13

Draw a graph of $\{(x,y)|x + y \nless 2\}$. This is equivalent to $\{(x,y)|x + y > 2 \text{ or } x + y = 2\}$. The graph of this relation is the complement of the graph of the relation defined by $x + y < 2$, as shown in Figure 5-14-14.

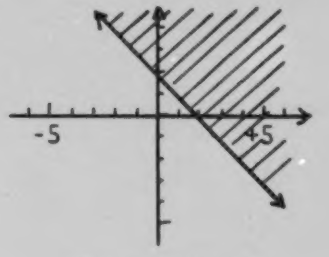


Figure 5-14-15

Exercises 5-14-16

Graph each of the following relations.

1. $\{(x,y)|y > -3x - 2 \text{ and } y \le -1/2x + 3\}$
2. $\{(x,y)|y \ge -3x - 2 \text{ or } y \le -1/2x + 3\}$
3. $\{(x,y)|y \le -3x - 2 \text{ and } y > x - 6\}$
4. $\{(x,y)|y > -3x - 2 \text{ or } y \ge x - 6\}$
5. $\{(x,y)|y \nless x - 6\}$
6. $\{(x,y)|y > x + 1\} \cap \{(x,y)|y < -2x - 3\}$
7. $\{(x,y)\ y > x + 1\} \cup \{(x,y)\ y < -2x - 3\}$
8. Write a program which will determine if a given ordered pair will satisfy the following inequalities:

   $$\{(x,y)|x + y < 2 \text{ and } x - y \le 4\}$$

9. Write a program to find all points whose coordinates satisfy:

   $$\{(x,y)|x + y < 2\} \cap \{(x,y)|\ 4x + y \ge -4\} \cap \{(x,y)|x \text{ and } y \text{ are integers}\}$$

10. Write a program which will find all points whose coordinates are integers between -5 and +5 and satisfy:

    $$\{(x,y)|x - y < 2\} \cup \{(x,y)|x + y < -4\}$$

11. Write a program which will find all points whose coordinates are integers between 0 and 5 and that satisfy:

    $$\{(x,y)|x - 2y \ge -12 \text{ and } 5x + 4y \le 40 \text{ and } x - 3y \le 24\}$$

Chapter 6

Circular Functions and Trigonometry

6-1 The Unit Circle

Several functions in the real numbers are derived from the properties of a circle. These functions are of considerable importance in the areas of electricity, mechanics and wave motion, as well as mathematics.

Definition 6-1-1 Unit Circle

> The circle with center at point (0,0), on the Cartesian plane, having a radius one unit in length is called the unit circle.

The unit circle is shown in Figure 6-1-2

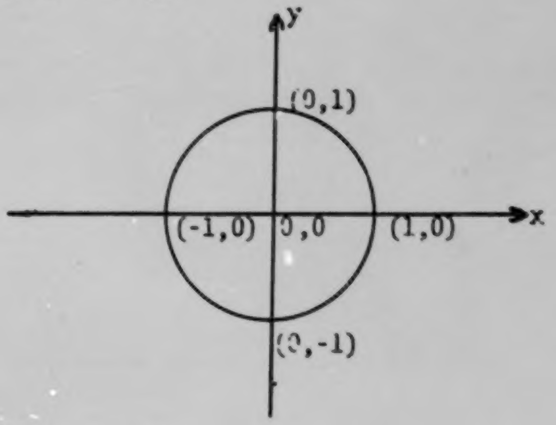


Figure 6-1-2

Given any point belonging to the unit circle, this point can be named by one of two methods. The first method is to state the real number which represents the distance from the point (1,0) to the given point as measured along the circumference of the unit circle. The second method of naming the given point is to state its Cartesian coordinates.

Example 6-1-3

Consider the point P_1, illustrated in Figure 6-1-4

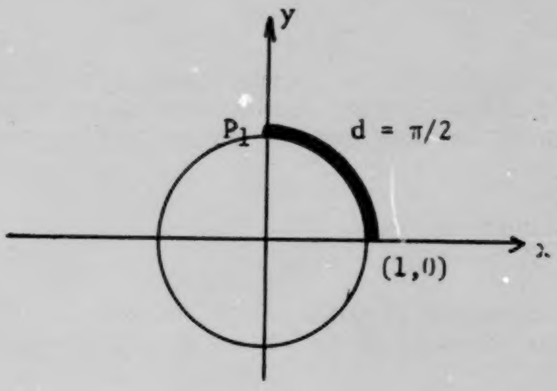


Figure 6-1-4

1. P_1 may be described by the real number, $d = \pi/2$, which is the distance from (1,0) to P_1 along the circumference of the unit circle.

and

2. P_1 may be described by the ordered pair (0,1)

How is this distance $d = \pi/2$ determined? Since the unit circle has a radius of one unit, its circumference is 2π. The distance from the point (1,0) to the point P_1 is 1/4 the circumference of the circle. Therefore,

$$d = 1/4 \cdot 2\pi = \pi/2$$

Example 6-1-5

Consider the point P_2, illustrated in Figure 6-1-6

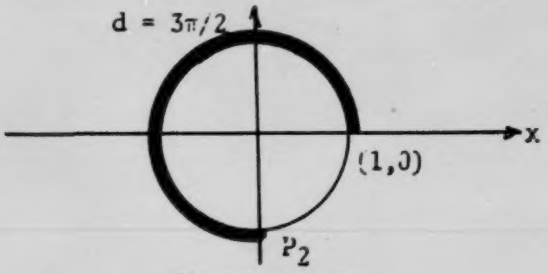


Figure 6-1-6

1. P_2 is described by $d = 3/4 \cdot 2\pi$ or $d = \frac{3\pi}{2}$ which is approximately 4.71

and

2. P_2 is described by the ordered pair (0,-1)

Example 6-1-7

Consider a point P shown in Figure 6-1-8

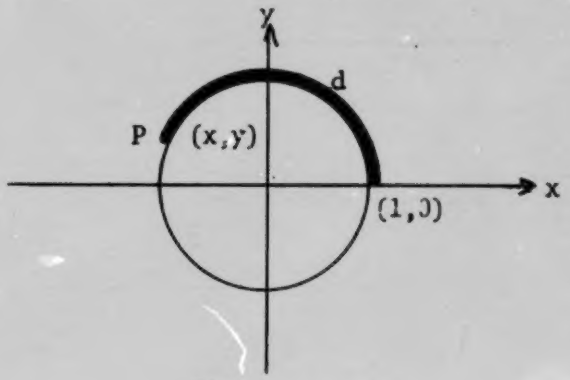


Figure 6-1-8

1. P could be described by the distance d measured along the circumference from (1,0).

2. P could be described by its cartesian coordinates (x,y).

In the previous examples, 6-1-3 and 6-1-5 we knew two exact values of d for two special points P_1 and P_2. Suppose d is any distance along the circumference of the unit circle in a counter-clockwise direction as shown in Figure 6-1-8. P(d) is the endpoint of the arc determined by (1,0) and d. It is important to realize that for each distance, d, there will be a unique point, P(d), called a trigonometric point.

The trigonometric point, P(d) will also be named by unique Cartesian coordinates (x,y). Therefore, for each distance d there exists a unique ordered pair (x,y) such that P(d) = (x,y).

6-2 Wrapping Function.

We will now define a new function P called the wrapping function.

Definition 6-2-1 Wrapping Function

$P = \{(d,(x,y)) \mid |d|$ is the distance along the unit circle from the cartesian point $(1,0)$ to the cartesian point $(x,y)\}$. When $d \geq 0, |d|$ is measured in a counter clockwise direction. When $d<0, |d|$ is measured in a clockwise direction. P is called the wrapping function.

The diagram in Figure 6-2-4 is an enla[illegible] view of the unit circle. It has been partially marked off in units corresponding to $(1/48 \cdot 2\pi)$ or $1/24 \cdot \pi$. We will use it to get approximations for some of the ordered pairs in the range of P, the wrapping function.

Example 6-2-2

Using Figure 6-2-4 find an approximation for $P(5\pi/12)$, to the nearest 0.01 units.

Moving a distance $d = 5\pi/12$, counter clockwise from $(1,0)$ along the unit circle, we arrive at a point whose coordinates are approximately $(0.26, 0.97)$

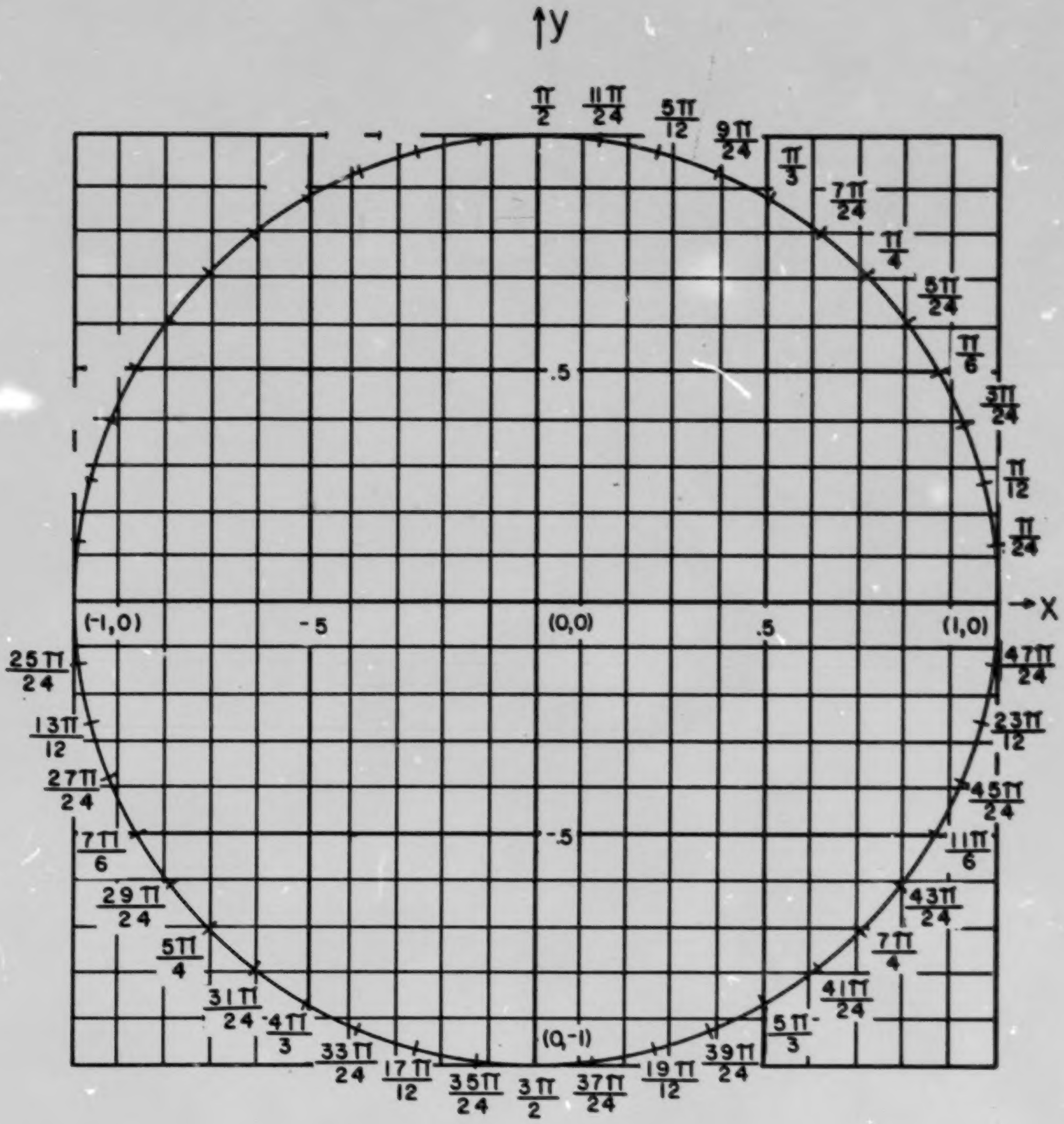
Therefore: $P(5\pi/12) \approx (0.26, 0.97)$

Example 6-2-3

Find an approximation for $P(-\frac{19\pi}{24})$, to the nearest 0.01 units from Figure 6-2-4

Moving a distance $|d| = \left|\frac{-19\pi}{24}\right|$, (clockwise) from the point $(1,0)$ along the unit circle, we arrive at a point whose coordinates are approximately $(-0.79, -0.61)$.

Therefore: $P(-19\pi/24) \approx (-0.79, -0.61)$



THE UNIT CIRCLE

Figure 6-2-4

Exercise 6-2-5

1. In the unit circle, which is illustrated in Figure 6-2-4 the names of the points, P(d), in the interval $\pi/2 < d < 25\pi/24$ have been omitted. Determine the real numbers, d, which name these trigonometric points and insert these on the figure.

2. Using Figure 6-2-4 complete the following table by approximating the coordinates of the ordered pairs which name P(d). The approximations should be to the nearest 0.01:

d	P(d)
(a) $\pi/6$	(0.87,0.49)
(b) $-11\pi/6$	
(c) $5\pi/24$	
(d) $-19\pi/24$	
(e) $5\pi/3$	
(f) $-\pi/3$	
(g) $11\pi/12$	
(h) $5\pi/4$	
(i) $-15\pi/24$	
(j) 2π	
(k) 2	
(l) -1	

3. Complete the following table by stating a value of d' so that P(d') is coincident with P(d) where d' and d have opposite signs.

	d	d'
(a)	$\pi/3$	$(-40/24)\pi$
(b)	$5\pi/6$	
(c)	$-13\pi/12$	
(d)	$7\pi/4$	
(e)	$-23\pi/24$	
(f)	$21\pi/12$	
(g)	$35\pi/24$	
(h)	$-19\pi/12$	

In Exercise 6-2-5 number 2, the distances, d, were all within the interval $0 < d < 2\pi$. If you check part (j) $p(-2\pi) = p(0) = p(2\pi)$. You may have assumed that this interval $-2\pi < d < 2\pi$ was the domain of the Wrapping Function. However, we shall use the set of real numbers as the domain of this function. Consider $d = 7\pi/3$. This is a complete circumference (2π) plus another $\pi/3$ distance into the second wrapping of the circumference. Checking Figure 6-2-4 we see that $P(7\pi/3) = P(\pi/3) \approx (0.50, 0.86)$.

Definition 6-2-6 The Domain of the Wrapping Function

The domain of the Wrapping Function is the set of all real numbers. These real numbers may be described as $\{x | x = 2k\pi + d \text{ where } k \varepsilon I, 0 \leq d \; 2\pi, \text{ and } d \varepsilon R\}$.

Exercise 6-2-7

1. Complete the following table by finding d and k. Approximate the coordinates of the ordered pairs which name P(x). The approximations should be to the nearest 0.01 from Figure 6-2-4

	$x = (2k\pi + d)$	k	d	P(x)
(a)	92π	46	0	(1, 0)
(b)	-37π			
(c)	$18\ \pi/4$			
(d)	$31\ \pi/6$			
(e)	$-7\ \pi/2$			
(f)	$29\ \pi/12$			
(g)	$50\ \pi/3$			
(h)	$15\ \pi/24$			
(i)	$-33\ \pi/4$			
(j)	$18\ \pi/6$			

2. Write a computer program to find d, $0 \leq d < 2\pi$ where $P(x) = P(d)$ and $x \varepsilon R$. Use the table in Exercise 1 for values of x. Use $\pi = 3.141592653$.

3. **Find a relationship between each of the ordered pairs:**

   (a) $P(d)$ and $P(d + 2\pi)$

   (b) $P(d)$ and $P(d + \pi)$

   (c) $P(d)$ and $P(d - \pi)$

   (d) $P(d + \pi)$ and $P(d - \pi)$.

4. Look at the set description of the domain of the wrapping function given in Definition 6-2-6 Do the restrictions $x = 2k\pi + d$, $0 \leq d \leq 2\pi$, $d \varepsilon R$ eliminate some real numbers from the domain?

6-3 Two Circular Functions - Sine and Cosine.

In the previous section we defined the domain of the Wrapping Function, P, as the set of real numbers. The range of P is a set of ordered pairs (x,y) where x and y each take on values within the intervals $-1 \leq x \leq 1$ and $-1 \leq y < 1$. Since P is a mapping of each real number onto a unique ordered pair of $\overline{R} \times R$ it is truly a function.

Now, we will state the definition of two of the six Circular functions.

Definition 6-3-1 The Sine of a Real Number

The sine of the real number x, abbreviated sin(x), is the second coordinate of the ordered pair which names the trigonometric point P(x).

Definition 6-3-2 The Cosine of a Real Number

The cosine of the real number x, abbreviated cos(x), is the first coordinate of the ordered pair which names the trigonometric point P(x).

From these two definitions we can describe the wrapping function as $P = \{(x,(\cos x,\sin x)) | x \varepsilon R\}$.

Let's now look at some examples of the sine and cosine of some real numbers.

Example 6-3-3

If $x = 3\pi/2$ find sin x and cos x.

The point $P(3\pi/2) = (0, -1)$

$\therefore \sin(3\pi/2) = -1$ (The second coordinate of $P(3\pi/2)$

and $\cos(3\pi/2) = 0$ (The first coordinate of $P(3\pi/2)$)

Example 6-3-4

If $x = -13\pi/12$ find the approximate values of sin x and cos x.

The point $P(-13\pi/12) \approx (-.97, .27)$ as shown in Figure 6-2-4

$\therefore \sin(-13\pi/12) \approx .27$

and $\cos(-13\pi/12) \approx -.97$

The coordinates of P(x) may be more precisely calculated than previously done by reading them from Figure 6-2-4. There are several methods of calculating the value of sin (x) and cos (x). The first method uses the properties of the unit circle (see Figure 6-3-5).

Suppose x is a given real number,

then $P(x) = (\cos(x), \sin(x))$.

According to the Pythagorean Theorem,

$\forall x, (\cos(x), \sin(x)) \in$ Unit Circle, $(\cos(x))^2 + (\sin(x))^2 = 1$. Why?

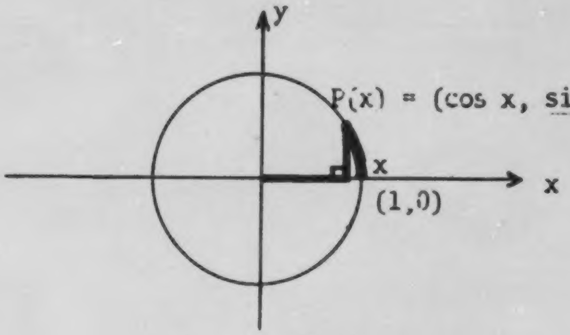


Figure 6-3-5

Hence, if one coordinate of the trigonometric point is known we can determine the other coordinate.

Example 6-3-6

Assume that Sin (x) = .3 for some real number x.

Since $(\cos(x))^2 + (\sin(x))^2 = 1$

then $(\cos(x))^2 + (.3)^2 = 1$

$(\cos(x))^2 = .91$

$|\cos(x)| = \sqrt{.91}$

or $|\cos(x)| \approx .95$

In conclusion, $\cos(x) \approx .95$ or $\cos(x) \approx -.95$ if $\sin(x) = .3$. The actual value is dependent upon which quadrant, I or II, contains the point P(x).

A method of approximating sine (x) and cos (x) for any number x is by the evaluation of one of the two infinite series:

$$\sin (x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \ldots$$

$$\cos (x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^{n+1} \frac{x^{2n-2}}{(2n-2)!} + \ldots$$

Theory of these infinite series cannot be presented at this point. However, the computer uses this method for determining the sine and cosine values for the real numbers.

We may determine the values for the sine or cosine of a real number through the use of the computer. Although these values are still approximations to the actual values, this method allows us to be precise to six digits. In order to access the approximate values of sin (x) and cos (x) from the computer, we can use any of the commands:

LET, PRINT, DEF, or IF _____ THEN _____ in the customary way.

Examples:

```
25 LET Y = SIN (8.7)

10  DEF FNX = COS (X)

30 PRINT SIN (X), COS (X)

85 IF SIN (X) = 1 THEN 110.
```

Another technique for finding sin (x) and cos (x) is to look them up in a table. You will find tables of values for sin (x) and cos (x), x expressed to the nearest .01, at the end of this chapter A sampling of these tables appears in Figure 6-3-7 These values, too, are approximate and were calculated by hand in early times.

```
      X                 APPRØX. X       SIN(X)              CØS(X)

-24 *PI/24              -3.14           0                   -1
-21 *PI/24              -2.75           -.38268             -.92388
-18 *PI/24              -2.36           -.70711             -.70711
-15 *PI/24              -1.96           -.92388             -.38268
-12 *PI/24              -1.57           -1                  0
-9 *PI/24               -1.18           -.92388              .38268
-6 *PI/24               -.79            -.70711              .70711
-3 *PI/24               -.39            -.38268              .92388
 0 *PI/24                0               0                   1
 3 *PI/24                .39             .38268              .92388
 6 *PI/24                .79             .70711              .70711
 9 *PI/24               1.18             .92388              .38268
 12 *PI/24              1.57            1                    0
 15 *PI/24              1.96             .92388             -.38268
 18 *PI/24              2.36             .70711             -.70711
 21 *PI/24              2.75             .38268             -.92388
 24 *PI/24              3.14            0                   -1
 27 *PI/24              3.53            -.38268             -.92388
 30 *PI/24              3.93            -.70711             -.70711
 33 *PI/24              4.32            -.92388             -.38268
 36 *PI/24              4.71            -1                   0
 39 *PI/24              5.11            -.92388              .38268
 42 *PI/24              5.5             -.70711              .70711
 45 *PI/24              5.89            -.38268              .92388
 48 *PI/24              6.28            0                    1
 51 *PI/24              6.68             .38268              .92388
 54 *PI/24              7.07             .70711              .70711
 57 *PI/24              7.46             .92388              .38268
 60 *PI/24              7.85            1                    0
 63 *PI/24              8.25             .92388             -.38268
 66 *PI/24              8.64             .70711             -.70711
 69 *PI/24              9.03             .38268             -.92388
 72 *PI/24              9.42            0                   -1

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Figure 6-3-7

Exercise 6-3-8

1. Look at Figure 6-3-7 These numbers are the computer values for the sine and cosine of some real numbers, x, whose approximate decimal equivalent is shown in the second column.

Complete the table below by following these instructions:

a. For each value of x in the table below, read the computer values of sin(x) and cos(x) from Figure 6-3-7 Enter these values in the column headed (cos(x), sin(x)).

b. Find each of the points (cos(x), sin(x)) on the Cartesian Plane shown in Figure 6-2-4

c. Find the distance along the unit circle in Figure 6-2-4 from the point (cos(x), sin(x)) to the point (1,0). Enter this distance in the table below as x_1.

d. In each case determine if $x = x_1$.

(cos x, sin x)	x_1	x	Does $x_1 = x$?
		$3\pi/24$	
		$9\pi/24$	
		$9\pi/8$	
		$-3\pi/8$	
		$23\pi/8$	

2. Utilizing all values for x and sin (x) shown in Figure 6-3-7 plot the graph of $\{(x,y) | y = \sin(x)\}$ on the x,y coordinate axes.

3. Consider the graph in Exercise 2. Answer the following questions about $\{(x,y) | y = \sin(x)\}$

   a. Is it a function?
   b. What is the apparent range?
   c. Does it appear to be continuous?
   d. What are the maximum and minimum points?
   e. Is there symmetry? Why or why not?
   f. Is this relation periodic? If so, what is the period?
   g. What are the intervals in which it is increasing? Decreasing?

In the exercise just completed, 6-3-8(3) we have discovered many properties of a relation which seems to be a function. Let us define it.

Definition 6-3-9 The Sine Function

The sine function, s, is defined by

$$s = \{(x,y)|y = \sin(x),\ x \varepsilon R\}$$

The domain of the sine function is defined to be the set of real numbers. From the initial definition of the sine of a real number and the properties of the unit circle we can see that the range of the function, defined by $y = \sin(x)$, is $\{y|\ -1 \leq y \leq +1\}$. This was also illustrated by the finite subset of s taken from Figure 6-3-7

Exercise 6-3-10

1. Using all values for x and cos (x) shown in Figure 6-3-7 plot the graph of $\{\overline{(x,y)}|y = \cos(x)\}$ on the x-y coordinate axes.

2. As in Exercise 6-3-8(3) answer the following questions about

$$\{(x,y)|y = \cos x,\ x \varepsilon R\}$$

   a. Is it a function?

   b. What is its apparent range?

   c. Does it appear to be continuous?

   d. What are the maximum and mimimum points?

   e. Is there symmetry? Why or why not?

   f. **Is this relation periodic? If so, what is the period?**

   g. What are the intervals in which it is increasing? Decreasing?

Definition 6-3-11 The Cosine Function

The cosine function, c, is defined by

$$c = \{(x,y) \mid y = \cos(x),\ x \varepsilon R\}$$

The domain of the cosine function is defined to be the set of real numbers. From the definition of cosine of a real number and the properties of the unit circle, we can see that the range of the function, defined by $y = \cos(x)$, is $\{y \mid -1 \leq y \leq +1\}$. This was illustrated in the previous Exercise 6-3-10-2

Exercise 6-3-12

Considering the two new functions, defined by $y = \sin(x)$ and $y = \cos(x)$, which of the following generalizations are true? If a generalization is not true, show a counter example. If a generalization seems to be true for any real number x, write a program designed to illustrate this fact with some examples.

Use $\pi = 3.141592653$

(a) $\forall_x \sin(x) = -\cos(x)$

(b) $\forall_x \sin(x) = \sin(-x)$

(c) $\forall_x \cos(x) = \cos(-x)$

(d) $\forall_x \sin(x) = -\sin(-x)$

(e) $\forall_x \sin(x) = \cos(\pi/2 - x)$

(f) $\forall_x \cos(x) = \sin(x + \pi/2)$

(g) $\forall_x \cos(x + \pi) = -\cos x$

(h) $\forall_x \sin(\pi - x) = \sin x$

6-4 Six Circular Functions

We will now define the four remaining circular functions. If you recall we defined the sine function (Definition 6-3-9) by defining what was meant by sin(x) for some real number (x) (Definition 6-3-1). Similarly, Definition 6-3-2, the cosine of a real number x, led us to the definition of the cosine function (Definition 6-3-11). Now, in a similar manner, we will define what is meant by four new terms, tangent, cotangent, secant and cosecant.

Definition 6-4-1

The tangent, abbreviated tan, of a real number x is defined by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \cos(x) \neq 0.$$

The cotangent, abbreviated cot, of a real number x is defined by

$$\cot(x) = \frac{\cos(x)}{\sin(x)}, \sin(x) \neq 0$$

The secant, abbreviated sec, of a real number x is defined by

$$\sec(x) = \frac{1}{\cos(x)}, \cos(x) \neq 0.$$

The cosecant, abbreviated csc, of a real number x is defined by

$$\csc(x) = \frac{1}{\sin(x)}, \sin(x) \neq 0$$

Exercise 6-4-2

1. For what values of x is tan(x) undefined?
2. For what values of x is cot(x) undefined?
3. For what values of x is sec(x) undefined?
4. For what values of x is csc(x) undefined?
5. Now that you think you have completed Exercises 1 - 4, above, have you considered all possible values for x?

From these definitions we may now define four new circular functions.

Definition 6-4-3

The Tangent Function:

$$t = \{(x,y) \mid y = \tan(x),\ x \neq \pi/2 + k\pi,\ k \in I\}$$

The Cotangent Function:

$$u = \{(x,y) \mid y = \cot(x),\ x \neq k\pi, k \in I\}$$

The Secant Function:

$v = \{(x,y) | y = \sec(x),\ x \neq \pi/2 + k\pi,\ k\epsilon I\}$

The Cosecant Function:

$w = \{(x,y) | y = \csc(x),\ x \neq k\pi,\ k\pi I\}$

Exercise 6-4-4

1. Write a program which will give the necessary information to plot the graphs of:

$\{(x,y) | y = \tan(x)\}$

$\{(x,y) | y = \cot(x)\}$

$\{(x,y) | y = \sec(x)\}$

$\{(x,y) | y = \csc(x)\}$

Use the definitions of these functions and the machine SIN(X) and COS(X) functions only. Plot the graphs.

2. From the graphs plotted in question 1 above and the exercises in the previous section. Complete the following table for the six circular functions.

	sine function s	cosine function c	tangent function t	cotangent function u	secant function v	cosecant function w
Domain						
Range						
Period						
Max Value						
Min Value						
Symmetry						
Points of Discontinuity						
Increasing Intervals						
Decreasing Intervals						

3. Solve the following equations:

   (a) $|\cos(x)| = 1$

   (b) $|\sin(x)| = 1$

   (c) $\tan(x) = 0$

   (d) $[\sin(x)] = -1$

6-5 Fundamental Identities

Now that we have defined the six circular functions, we are ready to study the inter-relationships between them. These relationships will be known as fundamental circular identities.

Our first task is to define what we mean by the term "identity."

Definition 6-5-1 Identity

The statement $f(x) = g(x)$ is an identity on the set S if and only if for the two functions f and g the following conditions hold:

(1) Domains D_f and D_g have common elements. That is,

$$D_f \cap D_g = S, \ S \neq \emptyset.$$

(2) $f(x) = g(x)$ for all x in S.

Example 6-5-2 The First Pythagorean Identity P_1

Prove that $(\sin(x))^2 + (\cos(x))^2 = 1$ is an identity.

To prove that this equation is an identity, we must show that both conditions (1) and (2) of Definition 5.51 are satisfied.

(1) Let $f(x) = (\sin(x))^2 + (\cos(x))^2$ and $g(x) = 1$

The domain of f is $D_f = \{x | x \epsilon R\}$

The domain of g is $D_g = \{x | x \epsilon R\}$

Hence $S = D_f \cap D_g = \{x | x \epsilon R\}$

Since S is a non-empty set, condition (1) of definition 5.51 is satisfied.

(2) For each $x \epsilon S$, $(\cos(x), \sin(x))$ are the coordinates of a point on the unit circle.

By the Pythagorean Theorem we know that $(\sin(x))^2 + (\cos(x))^2 = 1^2 = 1$

Since $f(x) = (\sin(x))^2 + (\cos(x))^2$ and $g(x) = 1$

We have shown that

$$\forall x \epsilon S, \ f(x) = g(x)$$

This satisfies condition (2) of Definition 6-5-1.

Since both conditions have been satisfied, the equation is an identity on the set S.

P_1: $\forall x, \sin^2 x + \cos^2 x = 1$

Notice that in the statement of this identity (P_1) we have written $(\sin(x))^2$ as $\sin^2 x$. It is customary to write the exponent above and to the right of the name of the function. It is the functional value which is squared or is raised to the exponent, not the value of x. This also allows the omission of the parentheses around the expression (sin(x)) and even those around (x) unless it is necessary for clarity.

We are now prepared to begin developing other identities.

Example 6-5-3 The Second Pythagorean Identity.

Prove the identity P_2

$$P_2: \forall x \neq k\pi, k \varepsilon I, \; 1 + \cot^2 x = \csc^2 x$$

(1) Let $f(x) = 1 + \cot^2 x$ and $g(x) = \csc^2 x$

The domain of f is $D_f = \{x | x \varepsilon R, x \neq k\pi, k \varepsilon I\}$

The domain of g is $D_g = \{x | x \varepsilon R, x \neq k\pi, k \varepsilon I\}$

Hence $S = D_f \cap D_g = \{x | x \varepsilon R, x \neq k\pi, k \varepsilon I\}$

The set S is non-empty and condition (1) is satisfied.

(2) In order to show that $1 + \cot^2 x = \csc^2 x$ is true for all $x \varepsilon S$, we begin with a statement which is known to be true. By application of valid reasoning we can then arrive at a true conclusion.

$$\sin^2 x + \cos^2 x = 1, \quad x \varepsilon R \qquad P_1$$

$$\left(\frac{1}{\sin^2 x}\right)(\sin^2 x + \cos^2 x) = \left(\frac{1}{\sin^2 x}\right)(1), \quad x \neq k\pi, k \varepsilon I \qquad \text{MPE}$$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \left(\frac{1}{\sin^2 x}\right)(1),\ x \neq k\pi,\ k\varepsilon I \qquad \text{DPMA}$$

$$1 + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x},\ x \neq k\pi,\ k\varepsilon I \qquad \text{RTPE}$$

$$1 + \cot^2 x = \csc^2 x,\ x \neq k\pi,\ k\varepsilon I \qquad \text{RTPE}$$

Therefore, the definition of identity is satisfied and

$$P_2:\ \forall x \neq k\pi,\ k\varepsilon I,\ 1 + \cot^2 x = \csc^2 x$$

From P_1, $\forall x$, $\sin^2 x + \cos^2 x = 1$ we can develop a third identity

$$P_3:\ \forall x \neq \pi/2 + k\pi,\ k\varepsilon I,\ \tan^2 x + 1 = \sec^2 x$$

Exercise 6-5-4

In a manner similar to the development in Example 6-5-3, prove the following identity from $\sin^2 x + \cos^2 x = 1$.

1. P_3: $\forall x \neq \pi/2 + k\pi,\ k\varepsilon I,\ \tan^2 x + 1 = \sec^2 x$

Prove each of the following:

2. $\forall x$, $\cos^2 x = 1 - \sin^2 x$

3. $\forall x \neq k\pi$, $k \varepsilon I$, $\cot^2 x = \csc^2 x - 1$

4. $\forall x \neq \pi/2 + k\pi$, $k \varepsilon I$, $1 = \sec^2 x - \tan^2 x$

The three identities

P_1: $\forall x$, $\sin^2 x + \cos^2 x = 1$

P_2: $\forall x \neq \pi/2 + k\pi$, $k \varepsilon I$, $\tan^2 x + 1 = \sec^2 x$

and P_3: $\forall x \neq k\pi$, $k \varepsilon I$, $\cot^2 x + 1 = \csc^2 x$

with their variations, are known as Pythagorean identities. This is due to the derivation of $\sin^2 x + \cos^2 x = 1$ from the properties of the unit circle and the application of the Pythagorean Theorem.

Another group of identities is called the reciprocal identities. Two of these identities depend upon the definition of the cosecant function.

Example 6-5-5 The First Reciprocal Identity

R_1: $\forall x \neq k\pi$, $k \varepsilon I$, $\csc x \cdot \sin x = 1$

(1) Let $f(x) = \csc x \cdot \sin x$ and $g(x) = 1$

$$D_f = \{x | x \varepsilon R, x \neq k\pi, k \varepsilon I\}$$

$$D_g = \{x | x \varepsilon R\}$$

Thus $S = D_f \cap D_g = \{x | x \varepsilon R, x \neq k\pi, k \varepsilon I\}$

(2) Again start with some appropriate true statement.

$$\csc x = \frac{1}{\sin x},\ x \neq k\pi,\ k\epsilon I \qquad \text{Def. csc x}$$

$$\sin x \cdot \csc x = \sin x \cdot \frac{1}{\sin x},\ x \neq k\pi,\ k\epsilon I \qquad \text{MPE}$$

$$\sin x \cdot \csc x = 1 \qquad x \neq k\pi,\ k\epsilon I \qquad \text{RTPE}$$

$$\csc x \cdot \sin x = 1 \qquad x \neq k\pi,\ k\epsilon I \qquad \text{CPA, RTPE}$$

Hence R_1 is an identity.

Example 6-5-6 The Second Reciprocal Identity

$$R_2: \quad \forall x \neq k\pi,\ k\epsilon I,\ \sin x = \frac{1}{\csc x}$$

(1) $S = \{x|x\epsilon R,\ x \neq k\pi,\ k\epsilon I\}$

(2) $\forall x \neq k\pi,\ k\epsilon I,\ \csc x \cdot \sin x = 1$ $\qquad R_1$

$$\left(\frac{1}{\csc x}\right)\cdot(\csc x \cdot \sin x) = \frac{1}{\csc x} \cdot 1 \qquad \text{MPE}$$

$$\sin x = \frac{1}{\csc x} \qquad \text{RTPE}$$

Now R_2: $\forall x \neq k\pi,\ k\epsilon I,\ \sin x = \frac{1}{\csc} x$ has been proved.

Two other identities are variations of the definition of the secant function:

$$\forall x \neq \pi/2 + k\pi,\ k\epsilon I,\ \sec x = \frac{1}{\cos} x.$$

R_3: $\forall x \neq \pi/2 + k\pi,\ k\epsilon I,\ \sec x \cdot \cos x = 1$

R_4: $\forall x \neq \pi/2 + k\pi,\ k\epsilon I,\ \cos x = \frac{1}{\sec x}$

From the definition of tan x,

$$\forall x \neq \pi/2 + k\pi,\ k\epsilon I,\ \tan x = \frac{\sin x}{\cos x},$$

and

$$\forall x \neq k\pi,\ k\epsilon I,\ \cot x = \frac{\cos x}{\sin x},$$

The following identities may be derived:

R_5: $\forall x \neq k \cdot \pi/2$, $k \varepsilon I$, $\tan x = \dfrac{1}{\cot x}$

R_6: $\forall x \neq k \cdot \pi/2$, $k \varepsilon I$, $\tan x \cdot \cot x = 1$

The identities P_1 - P_3 and R_1 - R_6 are known as the fundamental circular identities. These will be used in proving more complex relationships between circular functions.

Exercise 6-5-7

1. Derive R_3, R_4, R_5, and R_6 from the definition of the appropriate circular function. (Be certain to correctly restrict the domain.)

2. Complete the following table:

f(x)	g(x)	D_f *	D_g	$S = D_f \cap D_g$
1. sin x	cos x			
2. tan x	cot x			
3. sec x	csc x			
4. tan x	cos x			
5. csc x	sin x			
6. cot x	csc x			
7. tan x	sec x			
8. tan x	csc x			
9. cot x	sec x			
10. $\dfrac{\tan x}{\sec x}$	sin x			
11. tan x + cot x	sec x cscx			

---

* D_f represents Domain of the function f.

6-6

At this point you are familiar with the definitions of the six circular functions and, the Pythagorean and Reciprocal identities. We are ready to use them to prove more complicated identities. While there are many practical applications of identities, the reason we study them so thoroughly is their use in more advanced areas of mathematics, namely calculus.

In the following examples we will present and discuss many of the proper techniques for proving that an equation is an identity.

Example 6-6-1 Prove that

$$\forall x \neq k\,\pi/2,\ k \varepsilon I,\ \tan x + \cot x = \sec x \csc x.$$

(1) In Exercise 6-5-7 problem 2-11, we discovered that the intersection of the domains of the tangent, cotangent, secant and cosecant functions was $\{x | x \neq k\pi/2,\ k\varepsilon I\}$. Hence, this is the replacement set S over which $\tan x + \cot x = \sec x \cdot \csc x$ is possibly an identity.

Hence, we shall now show that:

$$\forall x,\ x \neq k\pi/2,\ k\varepsilon I,\ \tan x + \cot x = \sec x \csc x.$$

(2) $\forall x,\ x \neq k\pi/2,\ k\varepsilon I,$

$\tan x + \cot x = \tan x + \cot x$ RPE

$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$ RTPE

$\tan x + \cot x = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$ RTPE

$\tan x + \cot x = \frac{1}{\cos x \sin x}$ RTPE

$\tan x + \cot x = \frac{1}{\cos x} \cdot \frac{1}{\sin x}$ RTPE

$\tan x + \cot x = \sec x \cdot \csc x$ RTPE

Therefore:

$$\forall x,\ x \neq k\pi/2,\ k\varepsilon I,\ \tan x + \cot x = \sec x \cdot \csc x$$

Exercise 6-6-2

Complete the proof for the preceding example, starting with the statement:

$$\forall x,\ x \neq k\pi/2,\ k \varepsilon I,$$

$$\sec x \cdot \csc x = \sec x \cdot \csc x \qquad \text{RPE}$$

$$\vdots$$

$$\tan x + \cot x = \sec x \cdot \csc x$$

Example 6-6-3

State the replacement set, S, over which the following equation might be an identity and prove that it is.

$$\frac{\sin x}{\cos x \cdot \tan x} = \sin^2 x + \cos^2 x$$

(1) Establish S.

For the expression $\frac{\sin x}{\cos x \cdot \tan x}$, which real values of x must be excluded?

Since S is always a subset of the set of real numbers, we need to find those real numbers which when substituted for x is the expression $\frac{\sin x}{\cos x \cdot \tan x}$ or the expression $\sin^2 x + \cos^2 x$ do not yield names for real numbers. This will happen when substitution into $\cos x \cdot \tan x$ produces 0 or when $\tan x$ is undefined. Then $S = \{x | x \neq k\pi/2, k \varepsilon I\}$.

(2) Prove that the statement is true for all $x \varepsilon S$.

$\forall x,\ x \neq k\pi/2,\ k \varepsilon I$

$$\frac{\sin x}{\cos x \cdot \tan x} = \frac{\sin x}{\cos x \cdot \tan x} \qquad \text{RPE}$$

$$\frac{\sin x}{\cos x \cdot \tan x} = \frac{\sin x}{\cos x \cdot \frac{\sin x}{\cos x}} \qquad \text{RTPE}$$

$$\frac{\sin x}{\cos x \cdot \tan x} = \frac{\sin x}{\sin x} \qquad \text{RTPE}$$

$$\frac{\sin x}{\cos x \cdot \tan x} = 1 \qquad \text{RTPE}$$

$$\frac{\sin x}{\cos x \cdot \tan x} = \sin^2 x + \cos^2 x \qquad \text{RTPE}$$

Thus (1) and (2) show

$$\forall x,\ x \neq k\ \pi/2,\ k \varepsilon I \qquad \frac{\sin x}{\cos x \cdot \tan x} = \sin^2 x + \cos^2 x$$

is an identity.

A few general hints are in order to help you prove identities.

(1) Study definitions and fundamental identities.

(2) Be alert to such operations as squaring a binomial, factoring, and finding a least common multiple.

(3) If all other replacements fail, change expressions to their equivalents, in terms of the sine and cosine functions.

(4) Be imaginative.

(5) Have Patience!

Exercise 6-6-4

Supply the necessary restrictions on the replacement set of the variable and prove the following statements are identities:

1. $\sin x + (\cot x)(\cos x) = \csc x$
2. $(\cot a)(\sec^2 a) - \tan a = \cot a$
3. $\dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x}{\sin x}$
4. $\tan^2 t - \sin^2 t = (\tan^2 t)(\sin^2 t)$
5. $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$
6. $\dfrac{1 + \cot A}{1 + \tan A} = \cot A$
7. $\dfrac{(\cos^2 B)(\csc B)}{1 + \csc B} + \sin B = 1$
8. $\dfrac{\dfrac{\sin^3 x}{\cos x} + \sin x \cos x}{\tan x} = 1$
9. $\dfrac{1}{\csc x - \cot x} = \csc x + \cot x$
10. $(\tan s + \cot s)^2 = \sec^2 s \cdot \csc^2 s$
11. $\dfrac{1}{\sec^2 t} + \dfrac{1}{\csc^2 t} = 1$
12. $1 = 1/2[(\tan M)(\csc M)(\cos M) + (\cot M)(\sec M)(\sin M)]$
13. $(\csc^2 x)(\sec x) - (\cot x)(\csc x) + (\csc x)(\sec x) - \cot x = \sec x + \tan x$
14. $1/2\left(\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x}\right) = 1 + \tan^2 x$

6-7 Identities Continued

Up to this point in our study of identities we have been concerned with equations for circular functions which have arguments that consist of one term. We will now consider equations for circular functions with arguments containing more than one term. Expressions of the form $\cos(d_2 - d_1)$ and $\sin(d_2 - d_1)$ will be investigated and their relation to expressions of the form $\cos(d)$, and $\sin(d)$ will be developed. We will begin with the identity involving the cosine of the difference of two real numbers.

Identity D_1

$$\forall a, \forall b, \text{cox}(b - a) = (\cos b)(\cos a) + (\sin b)(\sin a)$$

Proof:

(1) Establish S and show that $S \neq \emptyset$

Let $f(a,b) = \cos(b - a)$

Let $g(a,b) = (\cos(b))(\cos(a)) + (\sin(b))(\sin(a))$

$D_f = \{(a,b) \mid a,b \in R\}$

$D_g = \{(a,b) \mid a,b \in R\}$

$\therefore S = D_f \cap D_g = \{(a,b) \mid a,b \in R\}$

(2) Prove that

$\forall a, \forall b \in S, \cos(b - a) = (\cos b)(\cos a) + (\sin b)(\sin a)$

Consider any two points $P(a)$ and $P(b)$ on the unit circle shown in Figure 6-7-1.

We can see by the definitions of sine and cosine that

$P(b) = (\cos b, \sin b)$

$P(a) = (\cos a, \sin a)$

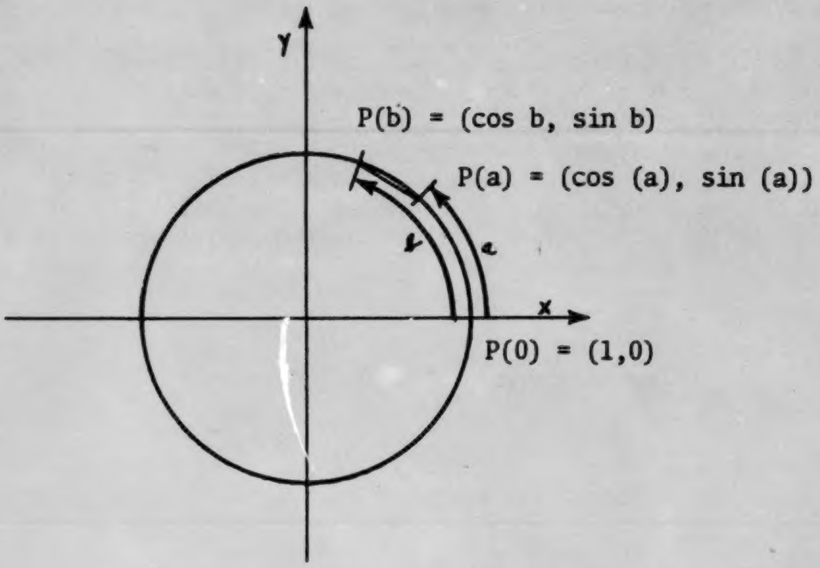


Figure 6-7-1

Using a form of the distance formula from Chapter 5, we can find an expression for the square of the distance between P(b) and P(a).

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$d^2 = (\cos b - \cos a)^2 + (\sin b - \sin a)^2$$

$$d^2 = \cos^2 b - 2(\cos b)(\cos a) + \cos^2 a + \sin^2 b - 2(\sin b)(\sin a) + \sin^2 a$$

$$d^2 = 1 - 2(\cos b)(\cos a) + 1 - 2(\sin b)(\sin a)$$

$$d^2 = 2 - 2(\cos b)(\cos a) - 2(\sin b)(\sin a)$$

We now want to derive another expression for d^2 involving cos(b - a). This is done by constructing a new set of coordinate axes, on the same unit circle as shown in Figure 6-7-2 on following page.

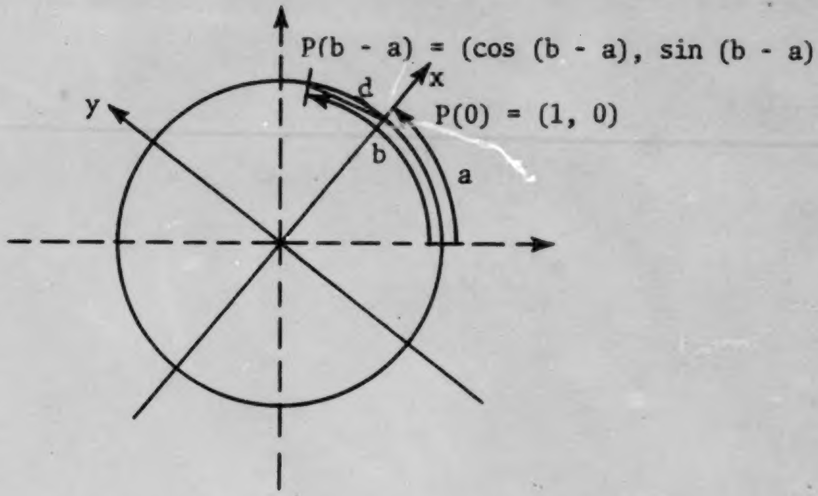


Figure 6-7-2

The distance between the same two points along the unit circle is now (b - a)

We can see by the definitions of sine and cosine that

$$P(b - a) = (\cos(b - a), \sin (b - a))$$

$$P(0) = (1,0)$$

While our coordinate axes are now different from those in Figure 6-7-1, the scale is the same and the relative position of the points has not been altered. Hence, the square of the distance between them is the same for each of the figures.

Using the same form of the distance formula, we can find an expression for the square of the distance between P(b - a) and P(0).

$$d^2 = (\cos(b - a) - 1)^2 + (\sin(b - a) - 0)^2$$

$$d^2 = \cos^2(b - a) - 2 \cos(b - a) + 1 + \sin^2(b - a)$$

$$d^2 = 1 - 2 \cos(b - a) + 1$$

$$d^2 = 2 - 2 \cos(b - a)$$

Since the distance, d, between P(d) and P(a), Figure 6-7-1, is equivalent to that distance, d, between P(b - a) and P(0), Figure 6-7-2, the squares of these distances, d^2, are also equivalent.

$$d^2 = d^2$$

$$2 - 2\cos(b - a) = 2 - 2(\cos b)(\cos a) - 2(\sin b)(\sin a)$$

$$-2\cos(b - a) = -2(\cos b)(\cos a) - 2(\sin b)(\sin a)$$

$$\cos(b - a) = (\cos b)(\cos a) + (\sin b)(\sin a)$$

Therefore, condition (2) of the definition of an identity is satisfied and

Identity D_1

$$\forall a, \forall b \cos(b - a) = \cos b \cos a + \sin b \sin a$$

is proved

From this point on, when proving an identity, we will not require the formal establishment of the replacement set S. However, the proof will be considered incomplete without an accurate description of the replacement set S.

We will find D_1 useful in proving that many other equations are identities. We start with an equation which was discussed previously.

Identity D_2

$$\forall a \cos(-a) = \cos(a)$$

Proof:

$$\begin{aligned}\cos(-a) &= \cos(-a)\\ &= \cos(0 - a)\\ &= (\cos 0)(\cos a) + (\sin 0)(\sin a)\\ &= 1 \cdot \cos a + 0 \cdot \sin a\\ &= \cos a\\ \therefore \forall a \cos(-a) &= \cos(a)\end{aligned}$$

Exercise 6-7-3

Prove the following identities:

Identity D_3 $\forall x, \cos(\pi/2 - x) = \sin x$

Identity D_4 $\forall x, \cos(x - \pi/2) = \sin x$

We will now prove identity D_5

Identity D_5

$$\forall a, \sin(\pi/2 - a) = \cos a$$

Proof:

Since we know nothing about $\sin(b - a)$ at this point, we must begin our proof with the statement

$$\cos a = \cos a$$

We now ask ourselves what form the variable, a, can be written in such that the expression $(\pi/2 - a)$ is introduced. Note that

$$a = (\pi/2 - (\pi/2 - a))$$

$$\begin{aligned} \cos a &= \cos a \\ &= \cos(\pi/2 - (\pi/2 - a)) \\ &= \cos \pi/2 \cos(\pi/2 - a) + \sin \pi/2 \cdot \sin(\pi/2 - a) \\ &= 0 \cdot \cos(\pi/2 - a) + 1 \cdot \sin(\pi/2 - a) \\ &= \sin(\pi/2 - a) \end{aligned}$$

$\therefore \forall a, \sin(\pi/2 - a) = \cos a$

We will now use identity D_5 to prove identity D_6.

Identity D_6

$$\forall a, \forall b, \sin(a + b) = (\sin a)(\cos b) + (\cos a)(\sin b)$$

Proof:

$$\begin{aligned}\sin(a + b) &= \sin(a + b)\\ &= \sin[\pi/2 - (\pi/2 - (a + b))]\\ &= \cos(\pi/2 - (a + b))\\ &= \cos((\pi/2 - a) - b)\\ &= \cos(\pi/2 - a)(\cos b) + \sin(\pi/2 - a)(\sin b)\\ &= \sin a \cos b + \cos a \sin b\end{aligned}$$

$$\therefore \forall a, \forall b, \sin(a + b) = \sin a \cos b + \cos a \sin b$$

Exercise 6-7-4

Prove the following identitiies

Identity D_7 $\quad \forall a, \sin(-a) = -\sin a$

Identity D_8 $\quad \forall a, \forall b \cos(a + b) = \cos a \cos b - \sin a \sin b$

Identity D_9 $\quad \forall a, \forall b, \sin(a - b) = \sin a \cos b - \cos a \sin b$

Exercise 5.73

Prove the following identities

(a) $\forall x, \sin(\pi - x) = \sin x$

(b) $\forall x, \cos(\pi - x) = -\cos x$

(c) $\forall x, \sin(x - \pi) = -\sin x$

(d) $\forall x, \cos(x - \pi) = -\cos x$

(e) $\forall x, \sin 2x = 2 \sin x \cos x$

(f) $\forall x, \cos 2x = \cos^2 x - \sin^2 x$

(g) $\forall x, \cos 2x = 2\cos^2 x - 1$

(h) $\forall x, \cos 2x = 1 - 2\sin^2 x$

Note: Identities e,f,g, and h are known as the identities of double angles.

Problem Set 6-7-5

The following identities may be proven using the identities previously developed in this chapter. Most of these are seen, in one form or another, throughout most trigonometry text books.

1. $\forall a, \sin^2(a) = \frac{1 - \cos(2a)}{2}$
2. $\forall a, \cos^2(a) = \frac{1 + \cos(2a)}{2}$
3. $\forall a, \forall b, a,b \neq k\,\pi/2, k \varepsilon I, \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$
4. $\forall a, \forall b, a,b, \neq k\,\pi/2, k \varepsilon I, \tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$
5. $\forall a, a \neq k\pi/2, k \varepsilon I, \tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$
6. $\forall a, a \neq k\pi/2, k \varepsilon I \tan a/2 = \sqrt{\frac{1 - \cos a}{1 + \cos a}}$ or $-\sqrt{\frac{1 - \cos a}{1 + \cos a}}$

   When is the positive sign appropriate? Negative sign?
7. $\forall a, a \neq k\pi/2, k \varepsilon I, \tan a/2 = \frac{1 - \cos a}{\sin a}$
8. $\forall a, a \neq k\pi/2, k \varepsilon I, \tan a/2 = \frac{\sin a}{1 + \cos a}$

For problems 9 - 20, determine the replacement set for the variable in addition to the proof of the identity.

9. $\sin 3x = 3 \sin x - 4 \sin^3 x$
10. $\cos 3x = 4 \cos^3 x - 3 \cos x$
11. $\sec 2x = 1/(2 \cos^2 x - 1)$
12. $2 \csc x = \cot x + \tan x$
13. $\cos 2x + 2 \sin^2 x = 1$
14. $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$
15. $\frac{\cos^2(x/2) - \cos x}{\sin^2(x/2)} = 1$
16. $\frac{2}{\cot y \tan 2y} = 1 - \tan^2 y$
17. $\frac{\cos^3 x - \sin^3 x}{\cos x - \sin x} = \frac{2 + \sin 2x}{2}$

18. $\frac{\sin x \tan(x/2)}{2} = \sin^2(x/2)$

19. $\sin x - \sin y = 2 \cos\left(\frac{x + y}{2}\right) \sin \frac{x - y}{2}$

20. $\tan \frac{1}{2} (A - B) = \frac{\sin (A - B)}{1 + \cos (A - B)}$

6-8 Period, Phase Shift, and Amplitude of Circular Functions

We are going to discuss functions defined by equations of the form

$$y = a \cdot f(b[x - c]) + d$$

where a, b, c, and d are real numbers and f represents one of the six circular functions. This discussion will begin with the function defined by

$$y = \sin x$$

and progress to functions defined by equations of the following forms.

(1) $y = a \sin x$

(2) $y = a \sin (b[x])$

(3) $y = a \sin (b[x - c])$

(4) $y = a \sin (b[x - c]) + d$

In each case we will analyze the graph of the function and the subsequent changes in the graph brought about by varying the values of a,b,c and d. For purposes of discussion we will analyze changes in the graph of $y = \sin x$. However, this analysis will be applicable to many of the other circular functions.

6-8-1 The Equation y = sin x

The equation $y = \sin x$ is equivalent to the equation $y = a \sin (b[x - c]) + d$ where $a = b = 1$ and $c = d = 0$. The graph of the function defined by this equation is shown in Figure 6-8-2

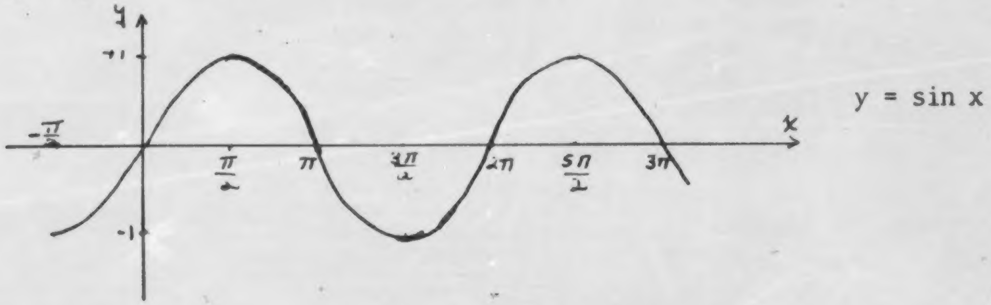


Figure 6-8-2

The period of this function is 2π as was discussed in a previous section. The maximum functional value is +1 and is called the amplitude of the function.

6-8-3 Equations of the form y = a sin x

For each value of a the equation y = a sin x defines a function. The similarities between graphs of these functions can be seen by sketching the graphs of y = 4 sin x, y = 1 sin x and y = (1/4) sin x on the same coordinate axes. We use the following computer program to produce a table of values for each function:

```
10 LET PI=3·141592653
20 PRINT "X", "APPROX. X", "4*SIN(X)", "1*SIN(X)", "(1/4)*SIN(X)"
30 FOR A=-24 TO 72 STEP 3
40 LET X=A*P1/24
50 PRINT A"*PI/"24,  X,  4*SIN(X),   1*SIN(X),    (1/4)*SIN(X)
60 NEXT A
70 END

RUNNH

X               APPROX. X      4*SIN(X)       1*SIN(X)        (1/4)*SIN(X)

-24 *PI/ 24     -3.14159       -2.51439E-9    -6.28598E-10    -1.57149E-10
-21 *P1/ 24     -2.74889       -1.53073       -.382683        -9.56709E-2
-18 *P1/ 24     -2.35619       -2.82843       -.707107        -.176777
-15 *P1/24      -1.9635        -3.69552       -.92388         -.23097
-12 *P1/ 24     -1.5708        -4             -1.              -.25
-9 *P1/ 24      -1.1781        -3.69552       -.92388          -.23097
-6 *P1/ 24      -.785398       -2.82843       -.707107        -.176777
-3 *P1/ 24      -.392699       -1.53073       -.382683         -9.56709E-2
 0 *P1/ 24       0              0              0                0
 3 *P1/ 24       .392699        1.53073        .382683         9.56709E-2
 6 *P1/ 24       .785398        2.82843        .707107          .176777
 9 *P1/ 24       1.1781         3.69552        .92388           .23097
 12 *P1/ 24      1.5708         4              1.               .25
 15 *P1/ 24      1.9635         3.69552        .92388           .23097
 18 *P1/ 24      2.35619        2.82843        .707107          .176777
 21 *P1/ 24      2.74889        1.53073        .382683         9.56709E-2
 24 *P1/ 24      3.14159        2.51439E-9     6.28598E-10      1.57149E-10
 27 *P1/ 24      3.53429       -1.53073       -.382683        -9.56709E-2
 30 *P1/ 24      3.92699       -2.82843       -.707107        -.176777
 33 *P1/ 24      4.31969       -3.69552       -.92388         -.23097
 36 *P1/ 24      4.71239       -4             -1.              -.25
 39 *P1/ 24      5.10509       -3.69552       -.92388         -.23097
 42 *P1/ 24      5.49779       -2.82843       -.707107        -.176777
 45 *P1/ 24      5.89049       -1.53073       -.382683        -9.56709E-2
 48 *P1/ 24      6.28319       -5.02878E-9    -1.2572E-9      -3.14299E-10
 51 *P1/ 24      6.67588        1.53073        .382683         9.56709E-2
 54 *P1/ 24      7.06858        2.82843        .707107          .176777
 57 *P1/ 24      7.46128        3.69552        .92388           .23097
 60 *P1/ 24      7.85398        4.             1.               .25
 63 *P1/ 24      8.24668        3.69552        .92388           .23097
 66 *P1/ 24      8.63938        2.82843        .707107          .176777
 69 *P1/ 24      9.03208        1.53073        .382683         9.56709E-2
 72 *P1/ 24      9.42478        7.68032E-9     1.92008E-9      4.8002E-10
```

A plot of these three functions may be seen in Figure 6-8-4

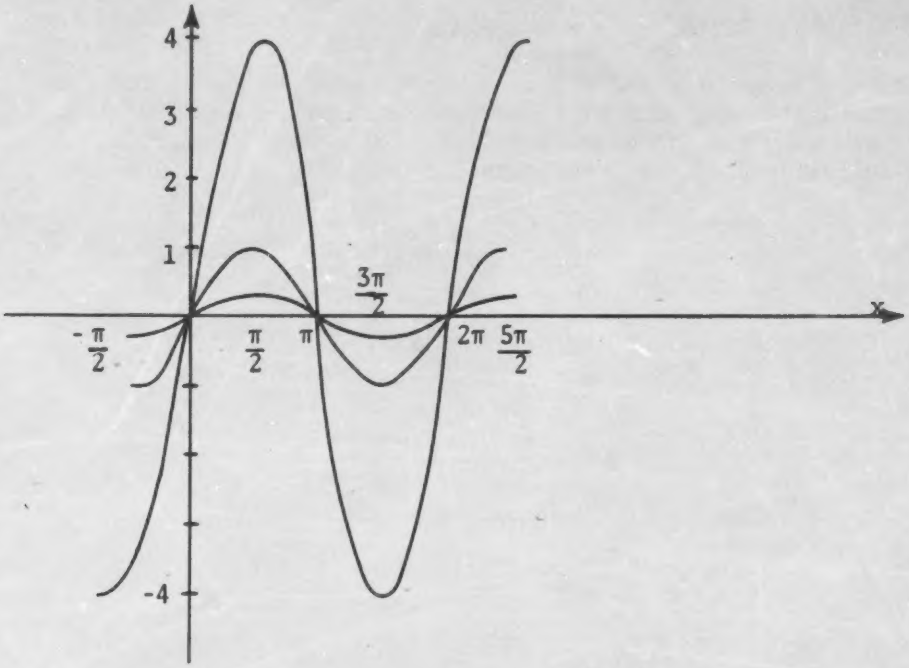


Figure 6-8-4

Notice that the range of the function $\{(x,y)|y = 4 \sin x\}$ is $\{y|-4 \le y \le 4\}$, while the range of $\{(x,y)|y - (1/4) \sin x\}$ is $\{y\ -1/4 \le y \le 1/4\}$. We notice a relationship between the particular values of a and the corresponding maximum and minimum functional values.

Definition 6-8-5 Amplitude of a Circular Function.

The amplitude of a function defined by

$$y = a \sin (b[x - c]) + d$$

or

$$y = a \cos (b[x - c]) + d, \quad a, b, c, d \in R \text{ is } |a|.$$

Example 6-8-6

Plot a graph of $\{(x,y) \mid y = (\frac{8}{5}) \sin x\}$ (Figure 6-8-7)

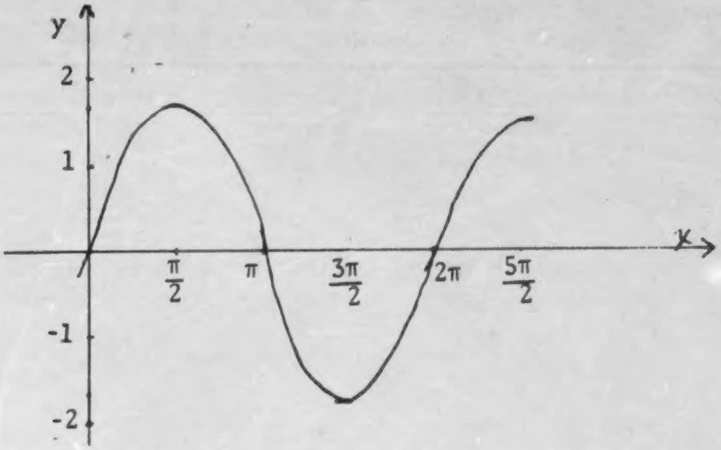


Figure 6-8-7

Exercise 6-8-8

1. For the following equations state the amplitude of the function and sketch its graph:
   - a. $y = 1/2 \sin x$
   - b. $y = 2 \sin x$
   - c. $y = -2 \sin x$
   - d. $y = -1/2 \sin x$
   - e. $y = 1/3 \cos x$
   - f. $y = -3 \cos x$
2. Graph the function defined by $y = a \sin x$, $a = 0$.
3. Discuss the difference between functions of the following forms:

   $R = \{(x,y) \mid y = a \sin x,\ a > 0\}$

   $R' = \{(x,y) \mid y = (-a) \sin x,\ a > 0\}$

6-8-9 Equations of the form $y = a \sin(bx)$

Careful examination of Figure 6-8-4 in the previous section suggests that all functions defined by equations of the form $y = a \sin x$ have the same period, namely 2π. We will now develop the effect of the real number b, $b > 0$ on the fundamental period of equations written in the form $y = a \sin(bx)$. Since we have defined a as a real number which determines the amplitude we simplify our initial discussions by considering cases for which $a = 1$.

Exercise 6-8-10

1. Write a program similar to the one shown in Section 6-8-3 and use it to generate the functional values for the following functions:

$$\{(x,y) \mid y = \sin(1/2\ x)\}$$

$$\{(x,y) \mid y = \sin x\}$$

$$\{(x,y) \mid y = \sin(3x)\}$$

2. Plot a graph of these three functions on the same set of coordinate axes.
3. What is the fundamental period of each of the three functions just plotted.
4. What regularities exist between the fundamental period of each function and the corresponding real number which replaces b in the general equation $y = \sin(bx)$?

Careful work with Exercise 6-8-10 reveals that for equations of the form $y = \sin(bx)$, the period appears to be $2\pi/b$. This is in fact the case. The function defined by $y = \sin x$ goes through one complete cycle, while $0 \leq x \leq 2\pi$. Any other equation of the form $y = \sin(bx)$ must also go through one complete cycle while

$$0 \leq bx < 2\pi \quad \text{or}$$

$$0 \leq x < 2\pi/b \qquad [\, b > 0 \,]$$

Hence, one cycle exists on the interval from 0 to $2\pi/b$ and the fundamental period of $y = \sin(bx)$ is $2\pi/b$.

This concept can be used to advantage when sketching graphs of circular functions, as seen in the following example.

Example 6-8-11

Sketch the graph of the equation

$$y = 3\cos(1/2\ x)$$

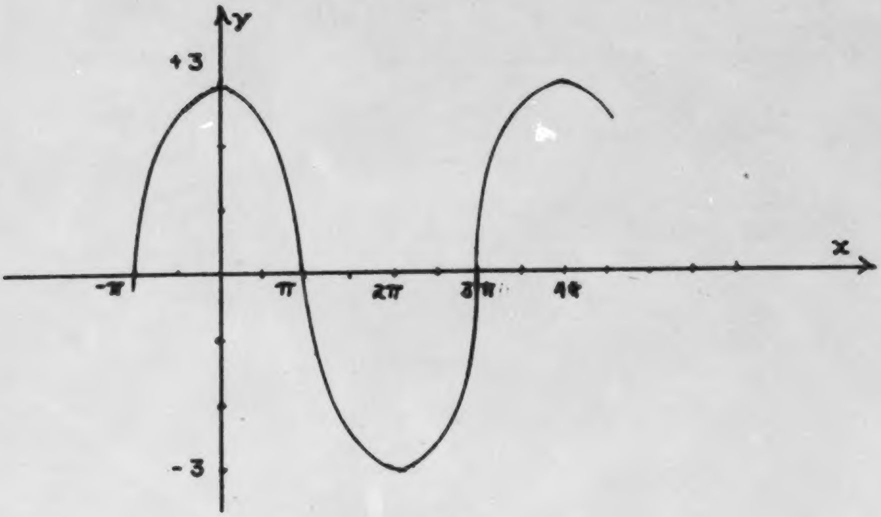


Figure 6-8-12

The graph of this equation is the same shape as that of $y = \cos x$ but the amplitude is $|3|$ and the period is $2\pi/(1/2) = 4\pi$

Exercise 6-8-13

Graph the following functions, stating the period of each and the amplitude if it exists:

1. $\{(x,y)|y = \sin(1/2)x\}$
2. $\{(x,y)|y = \sin 3x\}$
3. $\{(x,y)|y = \cos 4x\}$
4. $\{(x,y)|y = \cos(1/4)x\}$
5. $\{(x,y)|y = \tan 2x\}$
6. $\{(x,y)|y = 2\sin(1/4)x\}$
7. $\{(x,y)|y = 1/3\cos 2x\}$
8. $\{(x,y)|y = -3\sin(2\pi)x\}$
9. $\{(x,y)|y = 1/2\cos(\pi/4)x\}$
10. $\{(x,y)|y = 3\sec 2x\}$

6-8-14 Equations of the form $y = a \sin [b(x - c)]$

We shall consider any real number c and the effect it has on functions defined by equations of the form $y = a \sin [b(x - c)]$. This number, c, has no effect on either the amplitude or period of the function but it determines the horizontal position of the curve on the coordinate axes.

In order to simplify the situation we will begin with an example in which $a = b = 1$.

Example 6-8-15

Sketch the graph of the equation $y = \sin(x - 2)$

As we discussed in the previous section, the function defined by $y = \sin x$ completes one cycle while $0 \leq x < 2\pi$. Hence, the function defined by $y = \sin(x - 2)$ completes the same cycle while

$$0 \leq x - 2 < 2\pi \quad \text{or}$$

$$2 \leq x < 2\pi + 2$$

We conclude that the graph of $y = \sin(x - 2)$ is the same as that of $y = \sin x$ only the curve of $y = \sin x$ has been horizontally displaced 2 units to the right, along the x-axis. See Figure Figure 6-8-16 below. This horizontal displacement will be called phase shift.

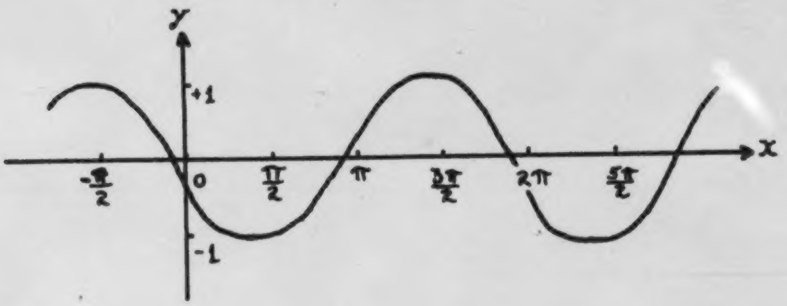


Figure 6-8-16

In general, for any function of the form $\{(x,y) | y = a \sin[b(x - c)]\}$. The same procedure may be used. That is, a function defined by any equation of the form $y = a \sin [b(x - c)]$ completes one cycle while $0 \leq b(x - c) < 2\pi$.

Hence, if $0 \leq b(x - c) < 2\pi$, for one cycle,

then $0 \leq x - c < 2\pi/b$ or

$c \leq x < 2\pi/b + c$.

This means that the amplitude of the equation $y = a \sin [b(x - c)]$ is $|a|$. The period is $2\pi/b$ and its phase shift is c. The curve will be shifted to the right if $c>0$ and to the left if $c < 0$. The curve is not shifted if $c = 0$.

Exercises 6-8-17

Sketch the graph of the following functions stating the period, phase shift, and amplitude, if it exists.

1. $\{(x,y) | y = \sin[3(x + 1)]\}$
2. $\{(x,y) | y = \sin (3x + 3)\}$
3. $\{(x,y) | y = \cos[2(x - \pi)]\}$
4. $\{(x,y) | y = \tan[1/2(x - 3)]\}$
5. $\{(x,y) | y = \sec[4(x - \pi/2)]\}$
6. $\{(x,y) | y = \cos(-1/3x + 3)\}$
7. $\{(x,y) | y = \tan(2\pi x + 8\pi)\}$

Example 6-8-18

Sketch the graph of the equation $y = 1/3 \cos (2x + 3)$

First, this equation must be written in the form $y = a \cos [b(x - c)]$,

Factoring the argument, $(2x + 3)$, we obtain

$$y = 1/3 \cos 2(x - (-3/2))$$

Hence, $|a| = |1/3| = 1/3$

$b = 2$

$c = -3/2$

The graph of this function can be obtained from the graph of $y = \cos x$ by

(1) reducing the amplitude to 1/3,

(2) reducing the period to $2\pi/2$ or π, and

(3) shifting the curve 3/2 units to the left along the x-axis.

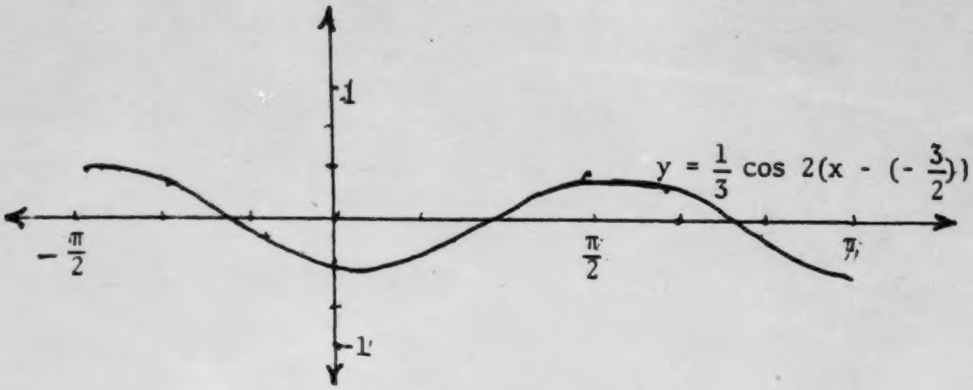


Figure 6-8-19

6-8-20 Equations of the form $y = a \sin [b(x - c)] + d$

With the addition of the term d to equations of the form previously studied, we obtain

$$y = a \sin [b(x - c)] + d$$

This is the last general equation to be studied. From the appearance of the equation one would think of this as defining a very complex function. However, just as the term c determined phase shift or horizontal displacement, the term d determines vertical displacement.

For ease of handling, let us consider the general equation for which $a = b = 1$ and $c = 0$, as in Example 6-8-21 below.

Example 6-8-21

Graph the equation $y = \sin(x) + 3$

Recall the graph of the curve defined by $f(x) = \sin x$. By substitution into $y = \sin(x) + 3$ we obtain $y = f(x) + 3$. Hence, for each value of x the functional value of $y = f(x) + 3$ will simply be three units greater than the functional value of $y = \sin x$. This is shown in Figure 6-8-22.

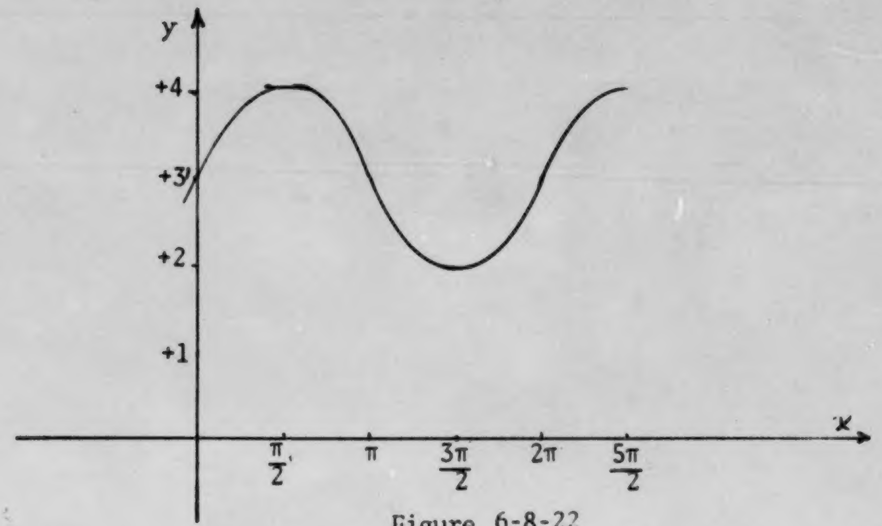


Figure 6-8-22

In general, the graph of any function of the form

$$\{(x,y) | y = a\ f(b[x - c]) + d\}$$

where f is a circular function,

can be sketched from the corresponding function

$\{(x,y) | y = f(x)\}$ by the following analysis.

(1) The amplitude, if it exists, will be $|a|$.

(2) The period will be p/b, where p is the fundamental period of the function f.

(3) The phase shift will be c units.

(4) The vertical displacement will be d units.

Example 6-8-23

Sketch the graph of the function$\{(x,y) | y = 4 \sin \pi/3[x - 2] - 1\}$.

(1) The amplitude is $|4|$ or 4.

(2) The period is $\frac{2\pi}{\pi/3}$ or 6.

(3) The phase shift is +2 units.

(4) The vertical displacement is (-1).

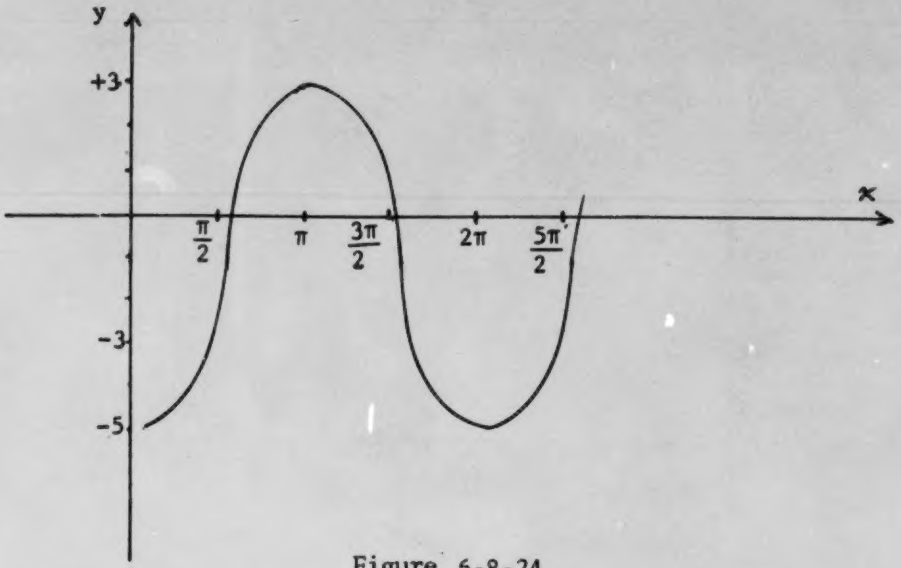


Figure 6-8-24

Exercise 6-8-25

Sketch the graph of each of the following functions and state the period, amplitude, and phase shift in each case.

$\{(x,y)\,|\,y = 2\cos(x + (\pi/2))\}$

$\{(x,y)\,|\,y = \cos(3x + \pi)\}$

$\{(x,y)\,|\,y = 4\cos(1/3\, x)\}$

$\{(x,y)\,|\,y = \sin(2x + \pi/2)\}$

$\{(x,y)\,|\,y = -1/2\sin(3x - \pi)\}$

$\{(x,y)\,|\,y = 3\cos(1/2x - \pi/3)\}$

$\{(x,y)\,|\,y = 2\sin(1/2\, x)\}$

$\{(x,y)\,|\,y = 4\sin(1/3x + \pi/6)\}$

6-9 Inverse Circular Functions

You are reminded that in Chapter 4 we defined the converse of a function to be a relation formed by interchanging the elements of each ordered pair of the function. If the converse is itself a function than we agreed to say it is an "inverse" of the original function.

Example 6-9-1

Let f = {(3,9), (7,8), (4,2), (8,3)} then the converse g = {(9,3), (8,7), (2,4), (3,8)} is the inverse of f because g is a function. Also, we adopted f^{-1} as a symbol to name the inverse of the function f. We will now relate the same concepts to the trigonometric functions.

Consider the square function (sq) which is the set of ordered pairs (x, x^2), $x \in R$. The graph of (sq) is shown in Figure 6-9-2. The coordinates of every point on this graph satisfy the equation $y = x^2$. If we form the converse of (sq) we get the set of ordered pairs which satisfy the equation $x = y^2$. The graph of this converse is also shown in Figure 6-9-2.

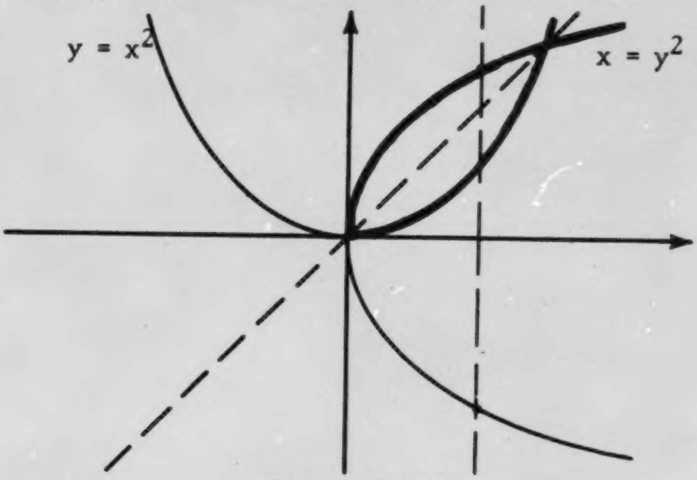


Figure 6-9-2

Applying the vertical line test described on page 4-23 we see that the relation $\{(x,y) | x = y^2\}$ is not a function. Hence, the inverse of (sq) does not exist.

The function (sq) is the union of two other functions (sq^+) and (sq^-) where $(sq^+) = \{(x,y) | y = x^2, x > 0\}$ and $(sq^-) = \{(x,y) | y = x^2, x < 0\}$. Each of the functions (sq^+) and (sq^-) has an inverse.

The graphs of sq^+ and its inverse rt^+ (rt. for root) are indicated by the heavy lines of Figure 6-9-2, those of sq^- and its inverse rt^-, by lighter lines. As you know, for every $x \geq 0$, $rt^+(x) = \sqrt{x}$, which is the principal square root of x, and $rt^-(x) = {}^-\sqrt{x}$. This example shows that by restricting the domain of a function, which does not have an inverse, we can derive new functions which do have inverses.

Now lets consider the sine function which is the set of ordered pairs (x, sin x). The graph is shown in Figure 6-9-3. The converse of the sine function which we will denote arcsin is the set of ordered pairs (sin x, x). The graph of this converse is shown by the broken line. The vertical line test shows that the converse arcsin is not a function, so the sine function does not have an inverse.

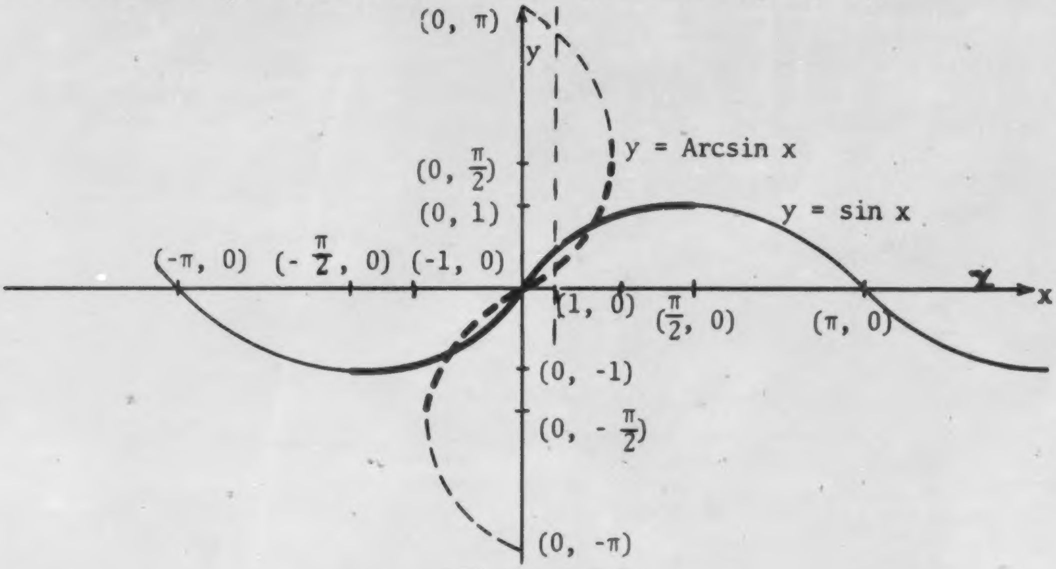


Figure 6-9-3

Since the range of the sine function is $-1 \leq y \leq 1$ we can restrict the domain to be $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and have a function that does have an inverse. (see the heavy portions of the graphs in Figure 6-9-3.) We will name this function, with restricted domain, Sin (read "cap sin" because of the capital ess.)

When we were considering the sq^+ function it was natural to call its inverse the rt^+ function. There isn't such an obvious name for the inverse of Sin. Remember that sine was an arbitrary name chosen for the function that associated the ordinate of any point on the unit circle to the arc distance the point was from the point (1,0). Therefore, we arbitrarily name the inverse function Arcsin, read "Arcsin" or "the inverse of Sin." Sin^{-1} is a symbol commonly used as a name for the Arcsin function. Sin^{-1} is also read "Arcsin" or "the inverse of Sin."

Definition 6-9-4

$$\text{Sin} = \{(x, \sin x) \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$$

$$\text{Arcsin} = \{(\sin x, x) \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$$

Example 6-9-5 Evaluate Arcsin (1)

Solution: To evaluate Arcsin (1) means to find the second element of the ordered pair (sin x, x) where the first element, sin x, is equal to 1, and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Hence, $x = \frac{\pi}{2}$.

Evaluate Arcsin $\frac{\sqrt{2}}{3}$

Solution: We need to find the second element of the ordered pair (sinx, x) where $\sin x = \frac{\sqrt{2}}{3}$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. $\frac{\sqrt{2}}{3} \approx \frac{1.414}{3} \approx 0.471$. From the table on page 6-97 we read that if sin x = 0.471, x = .489.

Exercise 6-9-6

Evaluate each expression.

1. $rt^{+}(16)$
2. $rt^{-}(16)$
3. $-rt^{+}(16)$
4. Arcsin $\frac{1}{2}$
5. Arcsin $\frac{\sqrt{2}}{2}$
6. Arcsin 0
7. Arcsin (-1)
8. Arcsin $(-\frac{\sqrt{2}}{2})$
9. Arcsin $(-\frac{\sqrt{3}}{2})$
10. Arcsin (sin π)
11. Arcsin (sin 1)
12. Arcsin (cos $\frac{\pi}{2}$)

Example 6-9-7 Solve $\sin \chi = \frac{1}{2}$ for χ.

Solution: $\arcsin \frac{1}{2} = \frac{\pi}{6}$ or $\frac{5}{6}\pi$ which are roots of the equation $\sin x = \frac{1}{2}$ But if we look at the graph of sin we can see that the equation has infinitely many solutions because $\frac{\pi}{6} + 2k\pi$ and $\frac{5\pi}{6} + 2k\pi$ are roots for each integer k. Therefore, the solution to our equation is:

$$x = \frac{\pi}{6} + 2k\pi, \text{ k an integer}$$
$$\text{or } \frac{5\pi}{6} + 2k\pi, \text{ k an integer}$$

Exercise 6-9-8 Solve each of the following equations.

1. $\sin x = 1$
2. $\sin x = -\frac{1}{2}$
3. $\sin x = 5$
4. $\sin y = \frac{\sqrt{3}}{2}$
5. $\sin t = 0$
6. $\sin \theta = \frac{1}{2}$
7. Arcsin x = 0
8. Arcsin a = 1
9. Arcsin b = 2

Like sin, the function cos does not have an inverse. (see the graph below, Figure 6-9-9.) Note that a line drawn parallel to the y axis, at some x, yields more than one second element, y, in the converse relation y = arccos x.

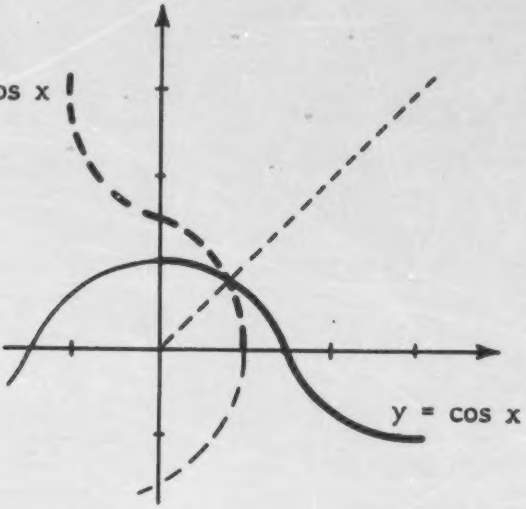


Figure 6-6-9

However, there are subsets of cos which do have inverses. The heavy portions of the above graph indicate one such subset. We will name this function Cos and its inverse Arccos.

Definition 6-9-10 Cos and Arccos

$$\text{Cos} = \{(x, \cos x) | 0 \leq x \leq \pi\}$$

$$\text{Arccos} = \{(\cos x, x) | 0 \leq x \leq \pi\}$$

Exercise 6-9-11

1. What are the domain and range of Cos? of Arccos?
2. Evaluate

(a) Arccos $\frac{1}{2}$ (b) Arccos $(-\frac{\sqrt{2}}{2})$ (c) Arccos 0

(d) Arccos (-1) (e) Arccos $(\frac{\sqrt{3}}{2})$ (f) Arccos $(-\frac{\sqrt{3}}{2})$

(g) Arccos (Cos $\frac{\pi}{2}$) (h) Arccos (Cos $(-\frac{\pi}{2})$) (i) Arccos (Sin $\frac{\pi}{3}$)

Like sin and cos, tan and cot do not themselves have inverses but each has subsets which have inverses. A subset of each, together with its inverse, is pictured below.

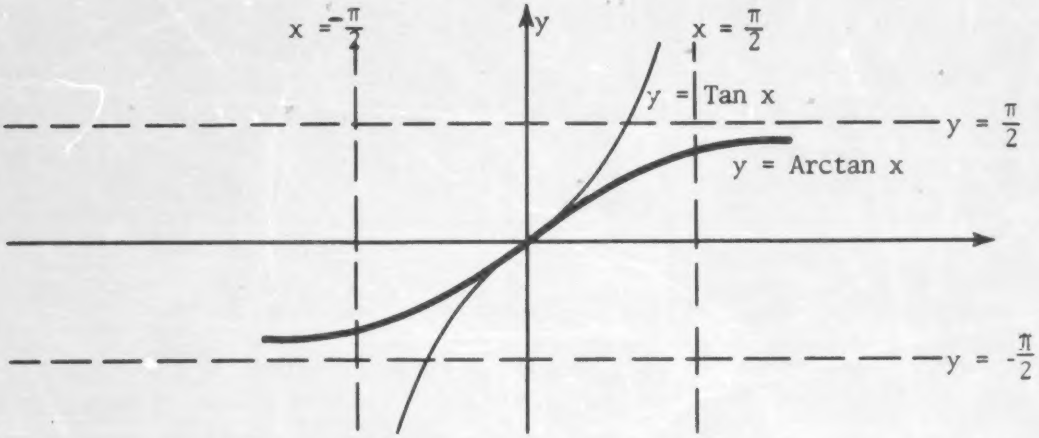


Figure 6-9-12

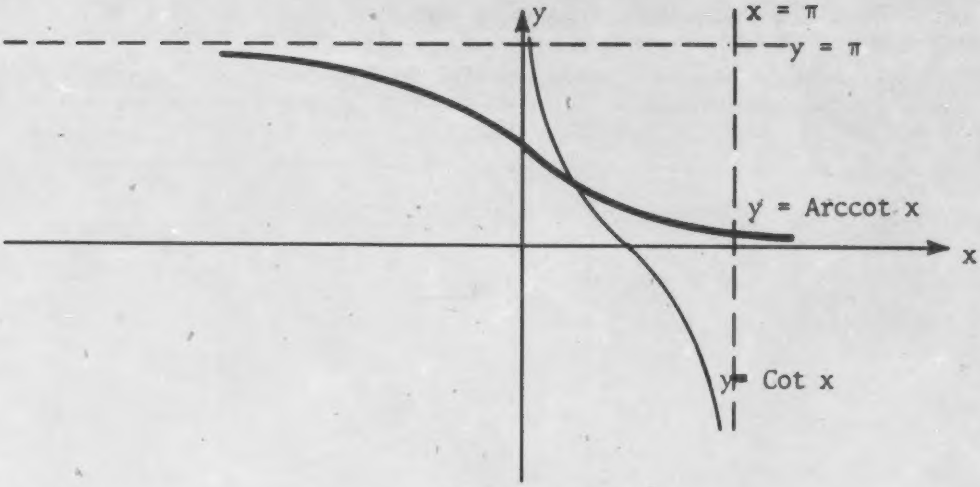


Figure 6-9-13

Definition 6-9-14 Tan, Arctan, Cot, Arccot

$$\text{Tan} = \{(x, \tan x) \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\}$$

$$\text{Arctan} = \{(\tan x, x) \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\}$$

$$\text{Cot} = \{(x, \cot x) \mid 0 < x < \pi\}$$

$$\text{Arccot} = \{(\cot x, x) \mid 0 < x < \pi\}$$

Exercise 6-9-15

1. What are the domain and range of Tan? of Arctan?
2. What are the domain and range of Cot? of Arccot?
3. Evaluate

(a) Arctan (-1) (b) Arccot 0 (c) Arctan $\sqrt{3}$

(d) Arccot (Sin $\frac{\pi}{6}$) (e) Arctan $\frac{\pi}{4}$ (f) Arctan (tan 5.796)

(g) Arccot (Arccos $\frac{\sqrt{3}}{2}$) (h) Arctan (Cot $\frac{\pi}{6}$) (i) Arccot (tan($-\frac{\pi}{4}$))

The concept of inverse functions is applied to solving of trigonometric equations as shown in the following examples.

Example 6-9-16 Solve 2 cos 4x = 1 for x

$$2\cos 4x = 1$$

$$\cos 4x = \frac{1}{2}$$

$$4x = \arccos\frac{1}{2} + 2n\pi,\ n \text{ an integer}$$

$$4x = \frac{\pi}{3} + 2n\pi,\ \text{or } \frac{5\pi}{3} + 2n\pi\ n \text{ an integer}$$

$$x = \frac{\pi}{12} + \frac{n\pi}{2},\ \text{or } \frac{5\pi}{12} + \frac{n\pi}{2}.\ n \text{ an integer}$$

Example 6-9-17 Solve $2\sin^2 y = 1$

$$\sin^2 y = \frac{1}{2}$$

$$\sin y = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

$$y = \arcsin\left(+\frac{1}{\sqrt{2}}\right) + 2n\pi,\ n \text{ an integer} = \frac{\pi}{4} + 2n\pi,\ \text{or } \frac{3\pi}{4} + 2n\pi\ n \text{ an integer}$$

$$y = \arcsin\left(-\frac{1}{\sqrt{2}}\right) + 2n\pi, n \text{ an integer} = -\frac{\pi}{4} + 2n\pi,\ \text{or } -\frac{3\pi}{4} + 2n\pi\ n \text{ an integer}$$

Example 6-9-18 Solve sin 2y - cos y = 0 for y

$$\sin 2y - \cos y = 0$$

$$2\cos y\sin y - \cos y = 0$$

$$\cos y\,(2\sin y - 1) = 0$$

$$\cos y = 0 \text{ or } 2\sin y - 1 = 0$$

$$\cos y = 0 \rightarrow \arccos 0 + 2n\pi = y$$

$$y = \pm\frac{\pi}{2} + n\pi,\ n \text{ an integer}$$

$$2\sin y - 1 = 0 \rightarrow \sin y = \frac{1}{2}$$

$$y = \arcsin\frac{1}{2} + 2n\pi,\ n \text{ an integer}$$

$$y = \frac{\pi}{6} + 2n\pi,\ n \text{ an integer}$$

$$\text{or} \qquad y = \frac{5\pi}{6} + 2n\pi,\ n \text{ an integer}$$

Exercise 6-9-19

Solve each of the following equations for x

1. $\tan x + \cot x = 2$
2. $\sin x + \sin 3x = \cos x$
3. $12\cos x - 5\sin x = 13$
4. $\cos 2x = \cos^2 x$
5. $\sin^2 x = \tan^2 x + 2$
6. $2\cos^2\left(\frac{x}{2}\right) + \cos 2x = 0$
7. $\sin 2x + \sin x - 2\cos x - 1 = 0$
8. $\sin x + \cos 2x = 0$
9. $\cos 3x - \sin x = \cos x$
10. $\tan 2x = 2\cos x$
11. $3 \cos x - 4 \sin x = 5$
12. $\tan x \cdot \tan 2x = 1$
13. $4 \cos x + 7 \sin x = 8$
14. $\sin 2x = \sqrt{2} \cos x$

6-10

Up to this point we have been concerned only with the six circular functions. Historically these functions were developed out of early work with a group of functions called trigonometric functions. We chose to study the circular functions first because of their simplicity and because they have subsets of the set of real numbers for their domains. We will now turn out attention to the trigonometric functions which have sets containing measures of angles for domains rather than sets of real numbers. You may have already seen these trigonometric functions in a Physics class. They have wide application in many other areas.

From work in geometry we realize that given any angle θ, we can construct the Cartesian axes in such a way that the origin is at the vertex of the angle and one of the sides lies along the positive x-axis. See Figure 6-10-1

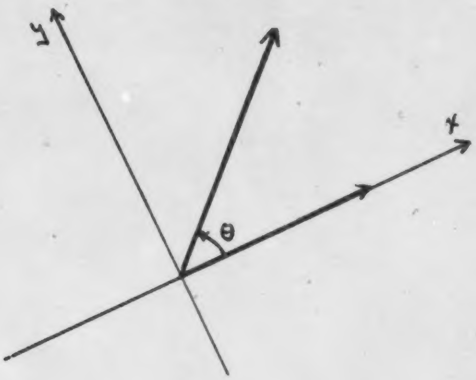


Figure 6-10-1

The angle θ is now said to be in standard position.

Now we may also construct a unit circle on the same coordinate system. See Figure 6-10-2

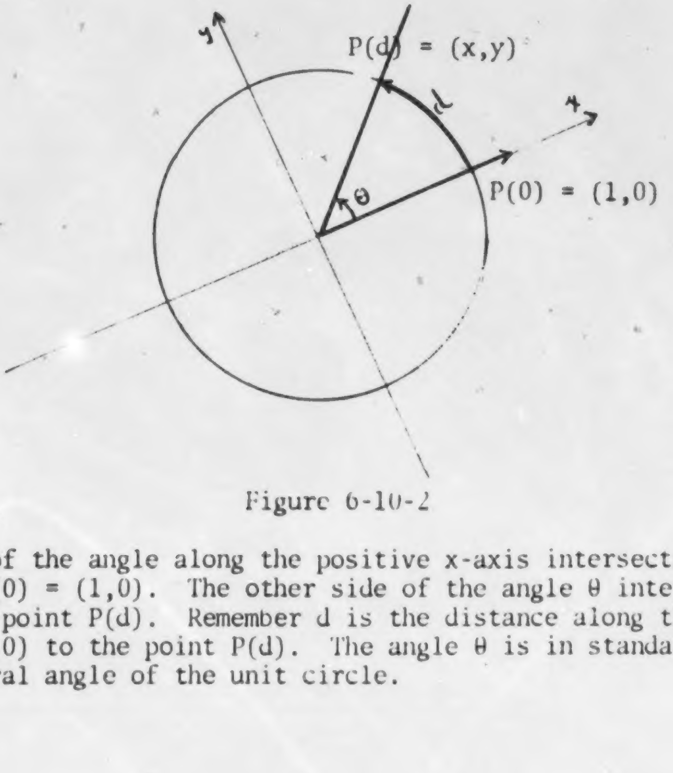


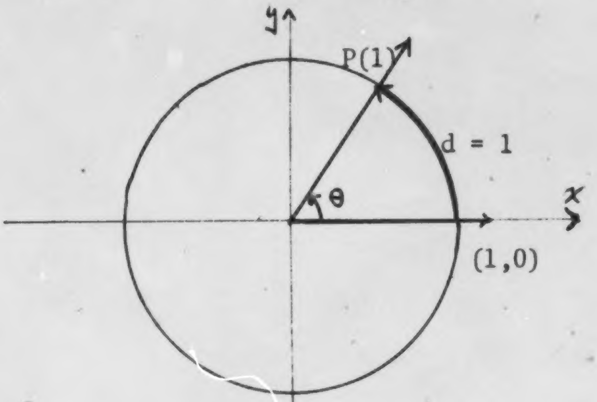
Figure 6-10-2

The side of the angle along the positive x-axis intersects the unit circle at the point P(0) = (1,0). The other side of the angle θ intersects the unit circle at some point P(d). Remember d is the distance along the unit circle from P(0) = (1,0) to the point P(d). The angle θ is in standard position and is also a central angle of the unit circle.

In geometry we learned that a central angle is measured by its intercepted arc. This means that the angle θ must have the same measure as the arc it intercepts. We can see that the measure of this arc is d units. Hence, the real number d is also a measure of the central angle θ. When this real number, d, is used as the measure of the central angle θ, this measure for θ is said to be in radians. Therefore, in Figure 6-10-2, θ = d radians.

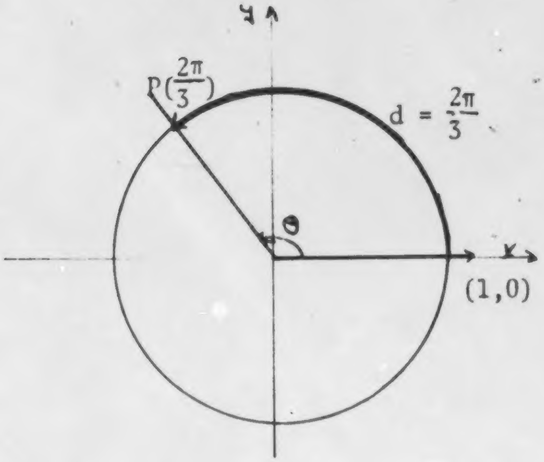
Definition 6-10-3 Radian

> A radian is the measure of a central angle in standard position which is subtended by an arc, on the unit circle, that is one unit in length.



θ = 1 radian

Figure 6-10-4

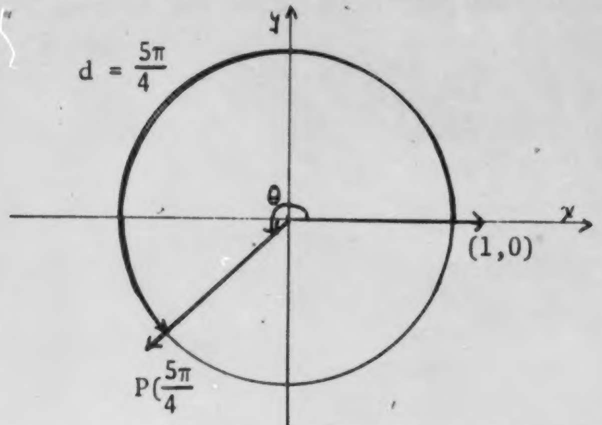


θ = 2 π/3 radians

Figure 6-10-5

If the angle θ is in standard position then $0 < \theta < \pi$ radians for in geometry we found that the measure of an angle was between 0 and 180 degrees. Since we want to discuss measures of θ beyond these limits we will simply refer to θ as an angle of rotation.

Consider Figure 6-10-6 where d = 5 π/4. The angle of rotation, θ is 5π/4 radians.

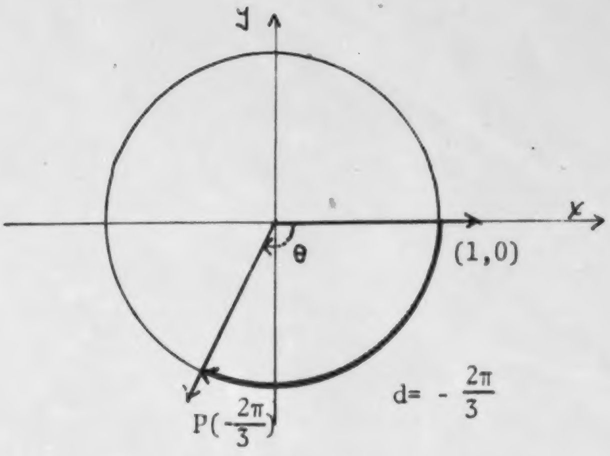


θ = 5 π/4 radians

Figure 6-10-6

Since we are establishing a correspondence between arc length along the unit circle and radian measure of the angle of rotation, we should remember that d, the arc length, can be less than zero. The convention which applies when d < 0 is that |d| is measured in a clockwise direction.

Consider Figure 6-10-7 where d = -2 π/3. The angle of rotation, θ, is -2 π/3 radians.



θ = -2 π/3 radians

Figure 6-10-7

Since we divide a circle into 360 parts, called degrees, angles of rotation may be measured in degrees or radians. Let's say that an angle, θ, has degree measure of 150°.

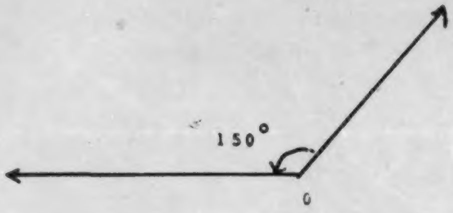


Figure 6-10-8

Placing this angle in standard position and superimposing a unit circle upon this axis,

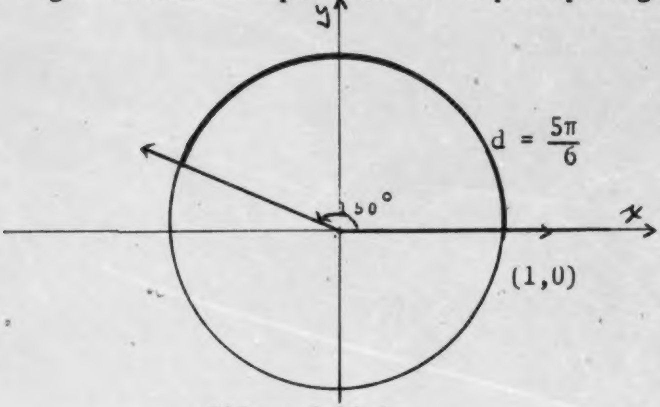


Figure 6-10-9

we see that

$$\frac{d}{2\pi} = \frac{150^{\circ}}{360^{\circ}}$$

$$d = \frac{150\pi}{180} = \frac{5\pi}{6}$$

Now $d = 5\pi/6$

and further $\theta = 150^{\circ} = 5\pi/6$ radians.

Example 6-10-10

Show that an angle of rotation with measure 225° is equivalent to 5 π/4 radians.

$$\frac{225^\circ}{360^\circ} = \frac{d}{2\pi}$$

$$d = \frac{225(2\pi)}{360} = \frac{5\pi}{4}$$

$$\theta = \frac{5\pi}{4} \text{ radians}$$

Example 6-10-11

Show that an angle of rotation with measure -11 π/6 radians is equivalent to -330°.

$$\frac{\theta}{360} = \frac{-11\ \pi/6}{2\pi}$$

$$\frac{\theta}{360} = \frac{-11\pi}{6} \cdot \frac{1}{2\pi}$$

$$\frac{\theta}{360} = \frac{-11}{12}$$

$$\theta = \frac{-11(360)^\circ}{12}$$

$$\theta = -330^\circ$$

Exercise 6-10-12

1. Convert the following angles of rotation from radian measure to degree measures.

a. π radians

b. π/2 radians

c. π/4 radians

d. - π/6 radians

e. 0 radians

f. 2π radians

g. -3 π/2 radians

h. 2π/3 radians

i. -2 π/2 radians

j. 5 π/6

k. $3\pi/4$ radians

l. -4π radians

m. $11\pi/12$ radians

n. $-4\pi/9$ radians

o. 3π radians (nearest tenth)

p. 1 radian (nearest tenth)

q. -2 radians (nearest tenth)

r. -3

2. Convert the following angles of rotation to radian measure.

a. 120°

b. 720°

c. 45°

d. -20°

e. 180°

f. 0°

g. 330°

h. 10°

i. 800°

j. -45

k. -90°

l. 360°

m. 1°

n. 13.5°

3. Each degree may be subdivided into 60 parts, each called a minute. Each minute may also be divided into 60 parts called seconds. Convert the following radian measures into degrees, minutes and seconds. (Round off to the nearest second).

a. 3.14 radians

b. .18 radians

c. 1 radian

Convert the following degree measures into radians. (Round off to the nearest one-hundreth of a radian).

d. $36^{\circ}\ 14'\ 22''$

e. $0^{\circ}\ 1'\ 0''$

f. $0^{\circ}\ 0'\ 1''$

4. a. Write a computer program to convert from degrees, minutes and seconds into radians.
   b. Write a computer program to convert from radians to degrees, minutes and seconds
   c. Run your programs with the data from Exercises 1 - 3 above.

We have seen that for every arc on the unit circle, measured by the real number d, there exists a unique angle of rotation, θ, such that

θ = d radians = $(\frac{360}{2\pi})$ · (d) degrees. We shall now define the six trigonometri functions. These functions are similar to the six circular functions except that their domains are sets containing measures of rotation rather than sets of real numbers.

Definition 6-10-13 The Trigonometric Functions

The trigonometric sine function, s_θ, is defined by

$$s_\theta = \{(\theta, \sin\theta) \mid \sin\theta = \sin d,\ \theta = d \text{ radians, or } \theta = (\tfrac{360}{2\pi})\ (d) \text{ degrees})\}$$

The trigonometric cosine function, c_θ, is defined by

$$c_\theta = \{(\theta, \cos\theta) \mid \cos\theta = \cos d,\ \theta = d \text{ radians, or } \theta = (\tfrac{360}{2\pi})(d) \text{ degrees}\}$$

The trigonometric tangent function, t_θ, is defined by

$$t_\theta = \{(\theta, \tan\theta) \mid \tan\theta = \tan d,\ \theta = d \text{ radians, or } \theta = (\tfrac{360}{2\pi})\ (d) \text{ degrees})\}$$

The trigonometric cotangent function, u_θ. is defined by

$$u_\theta = \{(\theta, \cot\theta) \mid \cot\theta = \cot d,\ \theta = d \text{ radians, or } \theta = (\tfrac{360}{2\pi})\ (d) \text{ degrees})\}$$

The trigonometric secant function, v_θ, is defined by

$$v_\theta = \{(\theta, \sec\theta) \mid \sec\theta = \sec d,\ \theta = d \text{ radians, or } \theta = (\tfrac{360}{2\pi})\ (d) \text{ degrees})\}$$

The trigonometric cosecant function w_θ, is defined by

$$w_\theta = \{(\theta, \csc\theta) \mid \csc\theta = \csc d,\ \theta = d \text{ radians, or } \theta = (\tfrac{360}{2\pi})\ (d) \text{ degrees})\}$$

To illustrate this correlation between the trigonometric functions and the circular function, let's look closely at the following figure.

Given, θ, an angle of rotation in standard position as in Figure 6-10-14

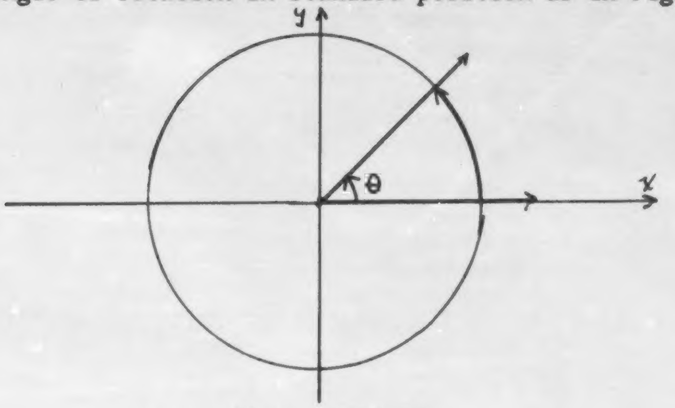


Figure 6-10-14

a. If $\theta = 75^{o}$ then sine θ, cos θ, tan θ, cot θ, sec θ, and csc θ all represent trigonometric functions.

b. If $\theta = 75^{o} = 5\ \pi/12$ radians then again sin θ, cos θ, tan θ, cot θ, sec θ, and csc θ all represent trigonometric functions.

However, if $\theta = 5\ \pi/12$ radians then $d = 5\ \pi/12$ and sin d, cos d, tan d, cot d, sec d, and csc d all represent circular functions.

The following exercise is designed as a quick check of your understanding of the difference between the trigonometric functions and the circular functions.

Exercise 6-10-15

Place a check (✓) in column 1 if the argument implies a trigonometric function. Check column 2 if the argument implies a circular function

Expression	(1) Trigonometric Function	(2) Circular Function
1. tan (72)		
2. sec (72^{o})		
3. sin (72 radians)		
4. cos (π/2)		
5. csc (-5 π/8 radians)		
6. tan (-24^{o})		
7. csc (154)		
8. cos (-17π)		
9. sin (-45 radians)		
10. sec (30^{o})		

Let us now consider an angle of rotation, θ radians, in standard position having a point P(x,y) on its terminal side which is a distance r from the origin. See figure 6-10-16.

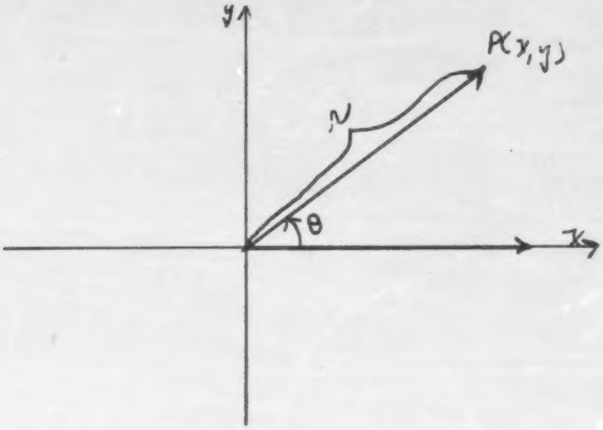


Figure 6-10-16

We construct a unit circle on these coordinate axes and, in addition, drop perpendiculars to the x-axis from the point P (x,y) and the point where the terminal side of the angle θ intersects the unit circle, as shown in Figure 6-10-17.

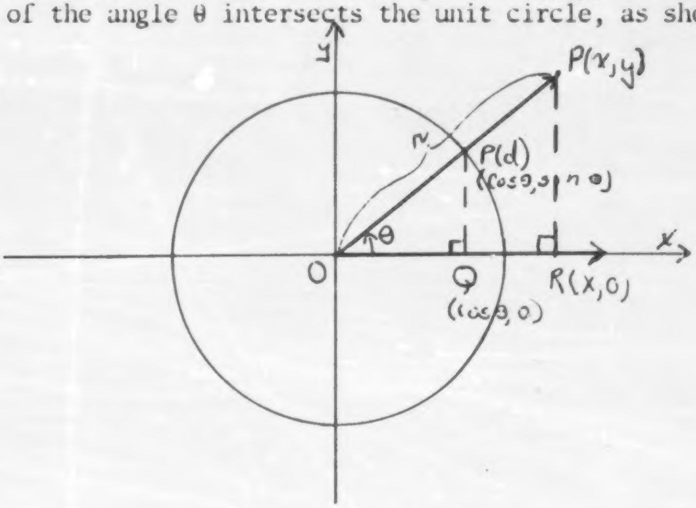


Figure 6-10-17

Then:

1. d is the real number θ.
2. P(d) has coordinates (cos d, sin d) or (cos θ, sin θ)
3. Point Q has coordinates (cos d, 0) or (cos θ, 0)
4. Point R has coordinates (x,0)
5. $\triangle\ 0\ Q\ P(d) \sim \triangle\ 0\ R\ P$

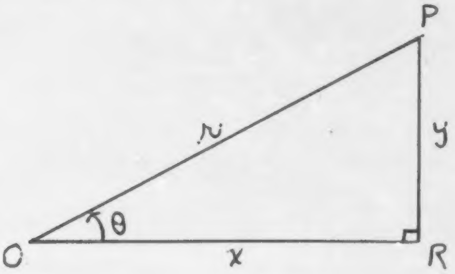
Now if Δ0 Q P(d)~Δ ORP

$$\text{then } \frac{x}{\cos\theta} = \frac{y}{\sin\theta} = \frac{r}{1}$$

so (1) $x = r\cos\theta$

(2) $y = r\sin\theta$

We can now redraw the angle θ and triangle OPR without the coordinate-axes or the unit circle.



$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

Figure 6-10-18

We have derived two equations which relate the measure of an angle of a right triangle to the measure of the sides of the triangle.

(1) $x = r\cos\theta \Longleftrightarrow \cos\theta = x/r,\ r \neq 0$

means that

In any right triangle with acute angle θ, the cosine of θ is determined by the ratio of the adjacent leg to the hypotenuse.

(2) $y = r\sin\theta \Longleftrightarrow \sin\theta = \frac{y}{r},\ r \neq 0$

means that

in any right triangle with acute angle θ, the sine of θ is determined by the ratio of the leg opposite the angle θ to the hypotenuse.

Example 6-10-19

Find sin θ, cos θ, sin α, cos α, for the following triangle ABC:

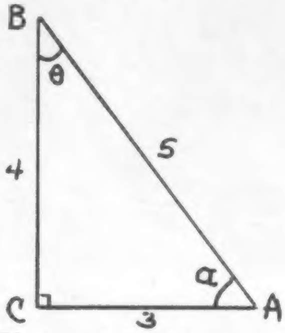


Figure 6-10-20

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{3}{5}$$

$$\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{4}{5}$$

$$\sin\alpha = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos\alpha = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{3}{5}$$

We will now develop a method for determining the value of the remaining four trigonometric functions of any acute angle θ of a right triangle.

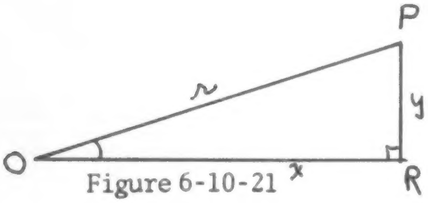


Figure 6-10-21

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$= \frac{y/r}{x/r}$$

$$= \frac{y}{x}$$

$$= \frac{\text{opposite leg}}{\text{adacent leg}}$$

Similarly,

$$\cot \theta = \frac{x}{y}$$

$$= \frac{\text{adjacent leg}}{\text{opposite leg}}$$

$$\sec \theta = \frac{r}{x}$$

$$= \frac{\text{hypotenuse}}{\text{adjacent leg}}$$

$$\csc \theta = \frac{r}{y}$$

$$= \frac{\text{hypotenuse}}{\text{opposite leg}}$$

Example 6-10-22

Consider the right triangle ABC, as shown in Figure 6-10-23.

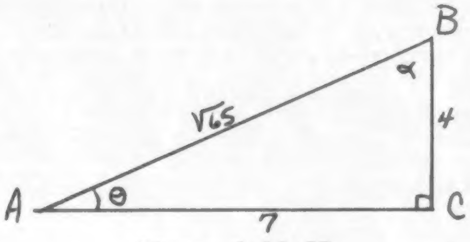


Figure 6-10-23

Find the values of the six trigonometric functions of θ.

$$\sin \theta = \frac{4}{\sqrt{65}} = \frac{4\sqrt{65}}{65} \approx .4961$$

$$\cos \theta = \frac{7}{\sqrt{65}} = \frac{7\sqrt{65}}{65} \approx .8682$$

$$\tan \theta = \frac{4}{7} \approx .5714$$

$$\cot \theta = \frac{7}{4} \approx 1.75$$

$$\sec \theta = \frac{\sqrt{65}}{7} \approx 1.152$$

$$\csc \theta = \frac{\sqrt{65}}{4} \approx 2.016$$

Find the values of the six trigonometric functions of α.

$$\sin \alpha = \frac{7}{\sqrt{65}} = \frac{7\sqrt{65}}{65} \approx .8682$$

$$\cos \alpha = \frac{4}{\sqrt{65}} = \frac{4\sqrt{65}}{65} \approx .4961$$

$$\tan \alpha = \frac{7}{4} = 1.75$$

$$\cot \alpha = \frac{4}{7} = .5714$$

$$\sec \alpha = \frac{\sqrt{65}}{4} \approx 2.016$$

$$\csc \alpha = \frac{\sqrt{65}}{7} \approx 1.152$$

Exercise 6-10-24

Find the values of the six trigonometric functions of the angle θ and the values of the trigonometric functions of the angle α, illustrated in the triangles below. Rationalize radicals which appear in the denominator.

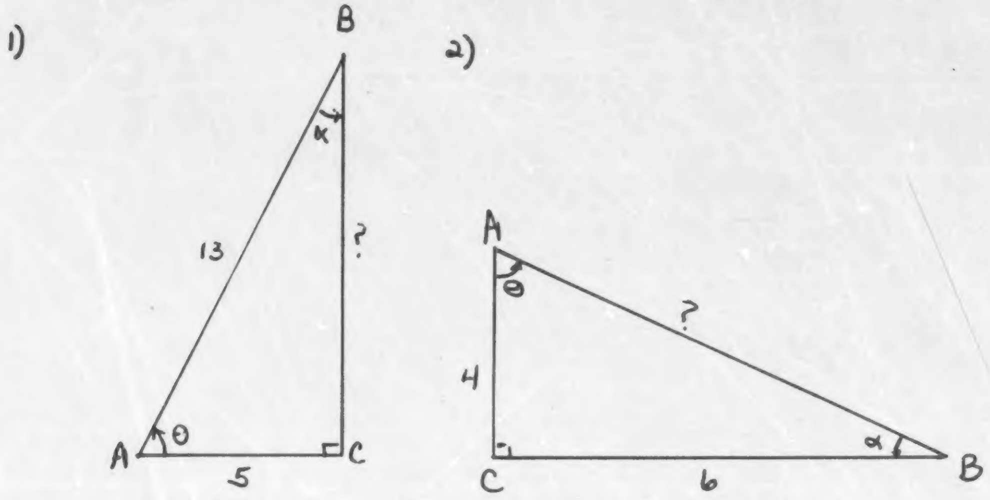


Figure 6-10-25

For a moment let us return to the properties of some special right triangles you discussed in geometry. If a right triangle has one acute angle whose measure is 30° or π/6 radians and the hypotenuse is one unit, the following facts may be stated:

(a) the side opposite the 30° angle is 1/2 unit long

(b) the side adjacent to the 30° angle is $\frac{\sqrt{3}}{2}$ units long (by Pythagorean Theorem)

(c) the sine of 30° is 1/2

(d) the cosine of $30^\circ = \frac{\sqrt{3}}{2}$

(e) the tangent of $30^\circ = \frac{1/2}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Exercise 6-10-26

Find the value of cot 30°, sec 30°, and csc 30°.

Exercise 6-10-27

Find the following:

1. sin 60° =
2. cos 60° =
3. tan 60° =
4. cot 60° =
5. sec 60° =
6. csc 60° =

In similar fashion look at an isosceles right triangle. The hypotenuse again is 1 unit. Each of the acute angles is equal to 45° or $\pi/4$ radians

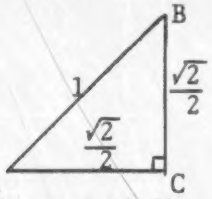


Figure 6-10-28

Using the Pythagorean Theorem we find that each of the legs has a measure of $\frac{\sqrt{2}}{2}$ units.

The sine of 45° is $\frac{\sqrt{2}}{2}$, cosine of 45° is $\frac{\sqrt{2}}{2}$ and the tangent of 45° is 1.

Exercise 6-10-29

Find

(1) cot 45°
(2) sec 45°
(3) csc 45°

Our knowledge of special triangles made it possible for us to evaluate the six trigonometric functions for the angles whose measures are 30°, 45°, and 60°. What if we wish to find the values of the trigonometric functions which correspond to angles having measures other than 30°, 45°, and 60°?

Given any acute angle, we can consider an unlimited number of right triangles containing that angle. These triangles would be similar triangles, the trigonometric ratios for each function would yield a constant value.

Example 6-10-30

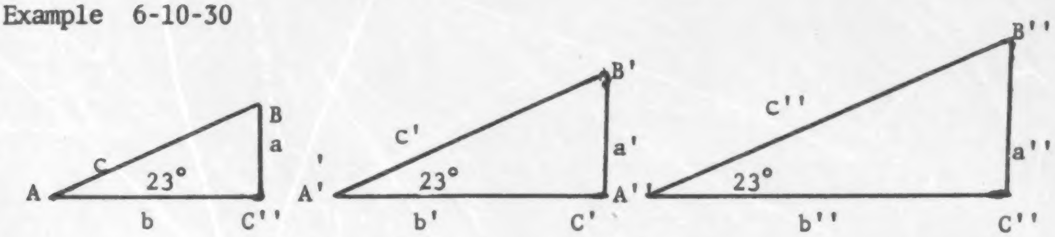


Figure 6-10-31

$\frac{a}{c} = \frac{a'}{c'} = \frac{a''}{c''}$ $\frac{b}{c} = \frac{b'}{c'} = \frac{b''}{c''}$ $\frac{a}{b} = \frac{a'}{b'} = \frac{a''}{b''}$

From Figure 6-10-31

$\sin 23° = \frac{a}{c}$ $\cos 23° = \frac{b}{c}$ $\tan 23° = \frac{a}{b}$

We could evaluate each ratio for 23° and all other possible measures of ∡A by this procedure and thereby build a list of the trigonometric function values for all possible angle measures. This is impractical, because there are an infinite number of angles to be considered. Even if we could consider all such angles, our ability to measure the sides accurately is questionable, therefore the accuracy of the ratio values would be in doubt.

Fortunately, tables of the trigonometric function values have been constructed by the use of formulas such as those stated on page 6-11.

6-11 Solution of Triangles

The earliest application of trigonometry in the history of man was to solve for unknown parts of a triangle. Man's need to solve right triangles brought about the development of the trigonometric functions much earlier than the concept of circular functions.

We will now see how the six trigonometric functions can be applied to solving right triangles. By the term "solving a triangle" we mean finding the measure of the unknown parts.

Example 6-11-1

A man wishes to know the distance between points A and C on opposite sides of a lake. (see figure 6-11-2 below) He walks from point C, at right angle to $\overline{AC}$ an arbitrary distance of 150 yards, to a point we will call B. With a surveyor's instrument (transit) he measures $\angle ABC$ to be 28°.

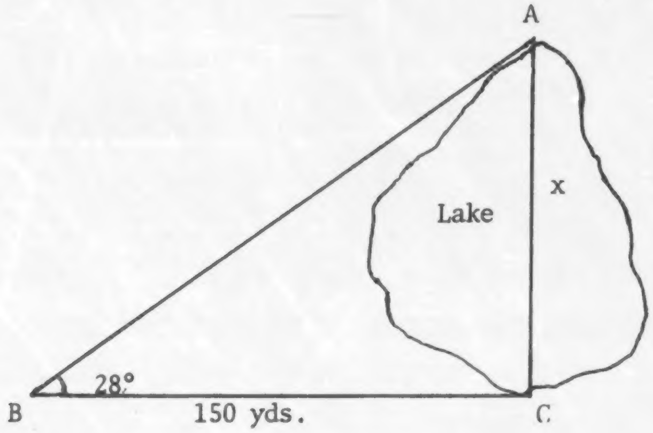


Figure 6-11-2

We will use the trigonometric function tan θ to find AC.

$$\tan \theta = \frac{\text{length of opposite leg}}{\text{length of adjacent leg}} = \frac{AC}{BC}$$

$\theta = 28°$, length of adjacent leg = 150, let length of opposite leg = x

$$\text{then } \tan 28° = \frac{x}{150}$$

$$x = 150\ (\tan 28°)$$

From the table of trigonometric function values we read tan 28° = .5317

$$\therefore\ x = 150\ (.5317)$$

$$x = 79.75 \text{ yards}$$

Example 6-11-3

We are asked to "solve" right triangle ABC, given c = 12 and A = 63°. That is, find all the unknown parts. We begin by sketching a right triangle approximately to scale.

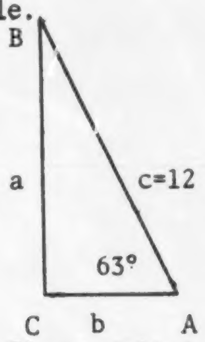


Figure 6-11-4

We must find ∠B and sides a and b

$$\sin A = \frac{\text{length side opposite}}{\text{length of hypotenuse}} \qquad \cos A = \frac{\text{length of side adjacent}}{\text{length of hypotenuse}}$$

$$\sin 63° = \frac{a}{12} \qquad \cos 63° = \frac{b}{12}$$

$$a = 12(\sin 63°) \qquad b = 12\cos 63°$$

$$a = 12(.8910) \qquad b = 12(.4540)$$

$$a = 10.7 \qquad b = 5.4$$

To find the measure of ∡B, we know:

$$A + B + C = 180°$$

$$63° + B + 90° = 180°$$

$$B = 27°$$

Example 6-11-5

Solve the right triangle ABC for "a" when $B = 47°16'$, $b = 10$

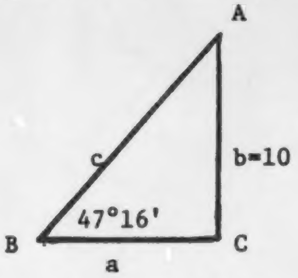


Figure 6-11-6

$$\text{We know that } \tan 47°16' = \frac{10}{a}$$

But this approach involves finding a quotient. A better approach would be:

$$\cot 47°16' = \frac{a}{10}$$

$$a = 10\,(\cot 47°16')$$

Now we are faced with evaluating cot 47°16' which we cannot read from our table. We can read from our table cot 47° and cot 48°, the value of cot 47°16' lies somewhere between these two values.

Let us look at the interval of the cot graph with which we are concerned.

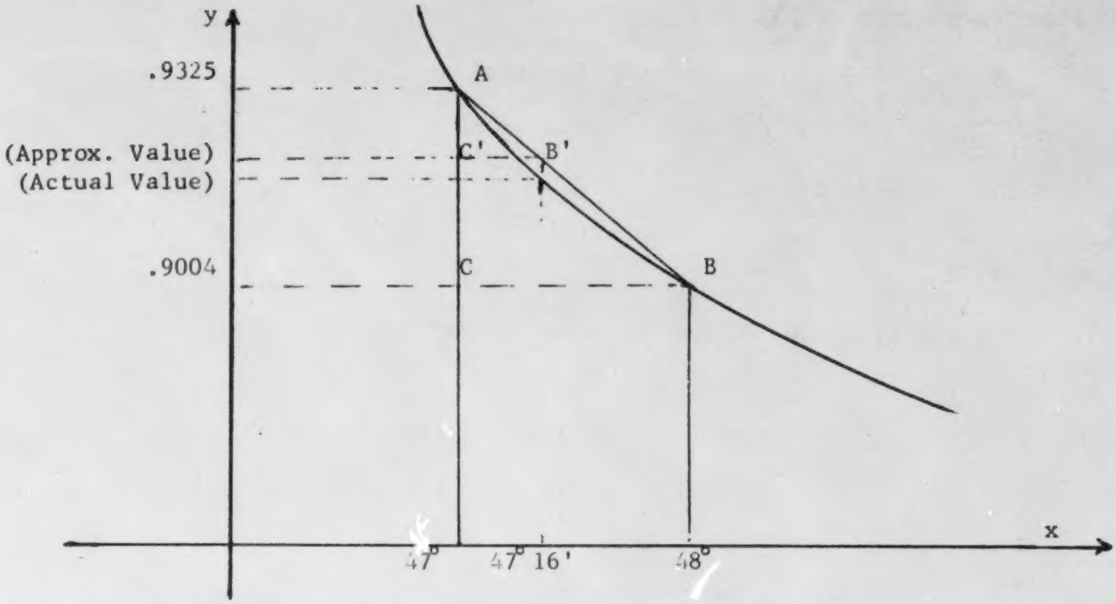


Figure 6-11-7

Triangles ABC and AB'C' are similar triangles, therefore

$$\frac{AC'}{AC} = \frac{B'C'}{BC}$$

From Figure 6-11-7 and the table of trigonometric functions:

$$AC = .9325 - .9004 = .0319$$

$$B'C' = 47°16' - 47° = 16'$$

$$BC = 48° - 47° = 1° = 60'$$

Substituting we have $\frac{AC'}{.0319} = \frac{16'}{60'}$

$$AC' = \frac{4}{15}(.0319) = \frac{.1276}{15} = .0085$$

Therefore, the approximate value of cot 47°16'

$$= \cot 47° - AC'$$

$$= .9325 - .0085$$

$$= .9240$$

This method of approximating the value of a trigonometric function is called linear interpolation. We approached the problem as though the values of the cot function formed a straight line graph. Since the cot function does not have a straight line graph our calculation produces an approximate value as illustrated.

Now to finish our problem from example 6-11-5.

(1) $a = 10\ (\cot 47°16')$

$a = 10\ (.9240)$

$a = 9.2$

If you understand the explanation above, the following short form to interpolation should be helpful and meaningful.

Example 6-11-8

In right triangle ABC find B, given a = 5, c = 11

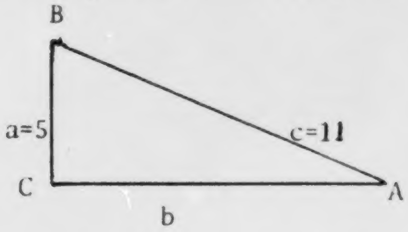
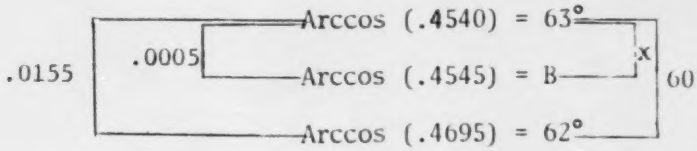


Figure 6-11-9

$$\cos B = \frac{5}{11} = .4545$$

*Arccos (.4545) = B

From our Trigonometric Function Value tables we determine that:



Arccos (.4540) = 63°

Arccos (.4545) = B

Arccos (.4695) = 62°

*The expression Arccos (.4545) is read, "The angle whose cosine value is .4545."

$$\frac{x}{60'} = \frac{.0005}{.0155}$$

$$.0155x = 60'(.0005)$$

$$x = \frac{60'(.0005)}{.0155} = 60' \cdot \frac{1}{31} = 1.9' \text{ (approximately 2')}$$

$$B = 63° - x = 62°58'$$

Figure 6-11-9 above indicates a method of labeling right triangles that has become generally accepted in mathematics literature. We will adopt the same method. Angle A will always be opposite side a, angle B opposite side b and angle C opposite side c.

Exercise 6-11-10

Solve the right triangle ABC, Given:

1. $a = 150$, $B = 31°$
2. $a = 250$, $b = 240$
3. $a = 192$, $A = 23°43'$
4. Now for a "practical" problem.

A stagecoach is traveling at 1150 feet per minute. A band of Indians is chasing the stagecoach at 1800 feet per minute. At the foot of a cliff 500 feet high is a stage-stop and safety. Two sentries standing on top of the cliff are watching the chase. They sight the angle of depression of the stage and find it to be 10°. The angle of depression of the band of Indians is 7° and of the stage-stop 70°. Can the Indians catch the stagecoach? Do the passengers gain the safety of the stage-stop? Calculate and see. (Illustration on page 6-77)

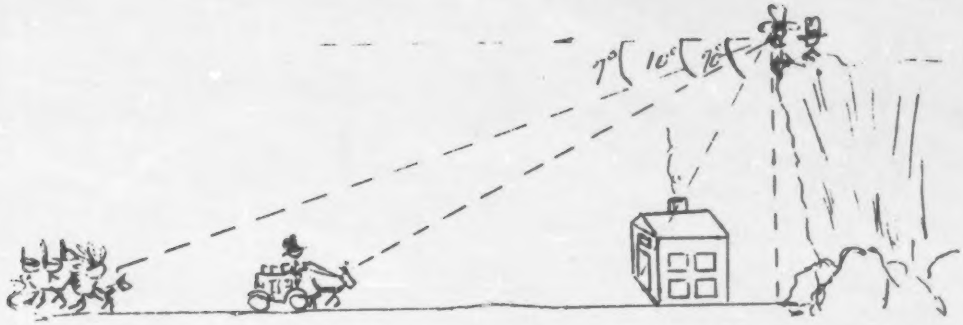


Figure 6-11-11

Example 6-11-12

Write a computer program to solve all right triangles given side a and ∠B as in problem 1 above.

We will use the formulas $\tan B = \frac{b}{a} \leftrightarrow b = a \tan B$,

$\cos B = \frac{c}{a} \leftrightarrow c = a \cos B$ and $A = 90° - B$.

Let A1 = side a, B1 = side b and C1 = side c.

10 INPUT A1, B

20 LET B1 = A1*TAN (B* $\frac{3.14159}{180}$)

30 LET C1 = A1*Cos (B* $\frac{3.14159}{180}$)

40 LET A = 3.14159/2 - B($\frac{3.14159}{180}$)

50 PRINT B1, C1, A* $\frac{180}{3.14159}$

60 END

Note that we use 3.14159 as an approximation for π. In lines 20 and 30 we convert degree measure to radian measure by multiplying B by $\frac{\pi}{180}$. In line 50 we convert the radian measure of A to degree measure.

Unfortunately, the BASIC language does not have the Arcsin or Arccos functions.

There is a simple way to overcome this problem.

Suppose $\sin x = b$. On the unit circle (see Figure 6-11-15) this means that the length of $\overline{PR}$ is b units.

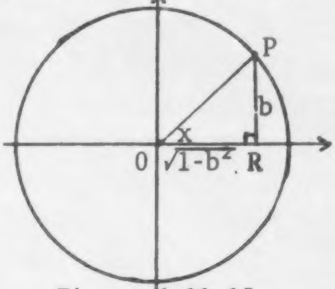


Figure 6-11-15

Now by the Pythagorean Theorem, we can compute the length of $\overline{OR}$ to be $\sqrt{1 - b^2}$. It can be seen now that $\tan x = b/\sqrt{1 - b^2}$ or $\text{Arctan}\,(b/\sqrt{1 - b^2}) = x$. Notice that the absolute value of b must be less than 1. Since we know the sine of x, which is b, we can now calculate the value of x.

A program to compute the value of x whose sine is B would be:

```
10  INPUT B
20  LET X = ATN(B/SQR(1 - B*B))
30  PRINT X "IS THE RADIAN MEASURE OF ANGLE X"
40  END
```

Similarly, if we were given the value of the cosine of x, to be equal to b, the tan of x will be equal to $\sqrt{1 - b^2}/b$. Then x will equal ATN $\left(\frac{\sqrt{1 - b^2}}{b}\right)$

Exercise 6-11-16

Write a computer program that will solve any right triangle, given two parts in addition to the right angle.

First we make a Macro flow chart of the process.

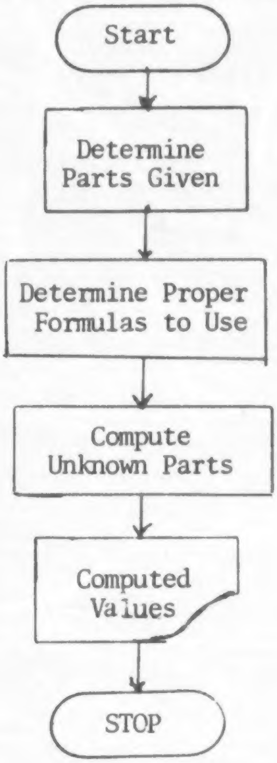


Figure 6-11-17

The determination of the given parts will give an indication as to which formulas to use in solving for the unknown parts.

We will use a coding process to tell the computer which parts are unknown.

Let us make a table that will show all the possible ways two parts out of the five unknown parts of a right triangle, could be given. We will use the following symbolism. A1, B1, and C1 will indicate the measures of the two legs and the hypotenuse respectively. A and B will represent the measure of the two acute angles. We will assign 0 (zero) to the parts whose measures are unknown. In our table the symbol K will indicate a known value.

CASE	A1	B1	C1	A	B
I	K	K	0	0	0
II	K	0	K	0	0
III	K	0	0	K	0
IV	K	0	0	0	K
V	0	K	K	0	0
VI	0	K	0	K	0
VII	0	K	0	0	K
VIII	0	0	K	K	0
IX	0	0	K	0	K
X	0	0	0	K	K

Figure 6-11-18

We see that there are 10 ways the known parts can be selected. The last case, given the two acute angles, would result in no unique triangle being determined.

We examine Case I, given sides A1 and B1.

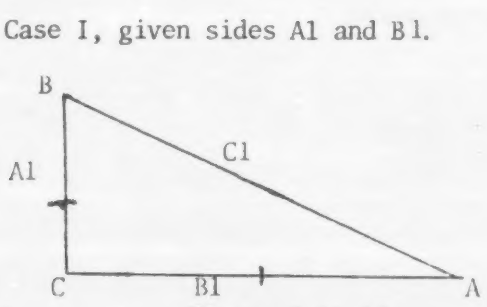


Figure 6-11-19

To find the unknown parts we use the following formulas.

$$C1 = \sqrt{(A1)^2 + (B1)^2} \qquad A = \text{Arctan}\,\frac{A1}{B1} \qquad B = \text{Arctan}\,\frac{B1}{A1}$$

Case II. Given sides A1 and C1

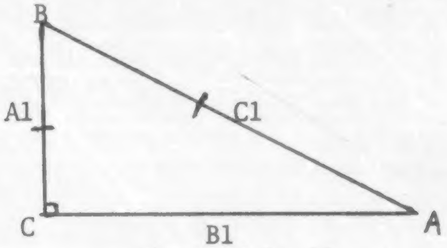


Figure 6-11-20

$$B1 = \sqrt{(C1)^2 - (A1)^2}$$

$$A = \text{Arcsin}\,\frac{A1}{C1} \qquad\qquad B = \text{Arccos}\,\frac{A1}{C1}$$

$$= {}^*\text{Arctan}\,\frac{A1}{B1} \qquad\qquad = {}^*\text{Arctan}\,\frac{B1}{A1}$$

$$= \text{Arctan}\ A1/\sqrt{(C1)^2 - (A1)^2} \qquad\qquad = \text{Arctan}\ \sqrt{(C1)^2-(A1)^2}\ /\ A1$$

Develop the necessary formulas for the remaining cases.

We will now write a program to solve any right triangle, given two of the five unknown parts. Remember that a "0" assigned to any part indicates that part is unknown, and must be computed. The angle measures are in degrees.

```
10 READ A1, B1, C1, A, B,

20 DATA 150, 0, 0, 0, 30, 192, 0, 0, 23.3, 0, 0, 0, 55, 0, 70.25,
        0, 49.6, 67.5, 0, 0

30 LET D = 3.14159
          .
          .
          .
```

*See Figure 6-11-20.

```
100  IF A1 <> 0 THEN 200
           .
           .
           .
200  IF B1 <> 0 THEN 300
           .
           .
           .
                       A1       180
300  PRINT "A="; ATN (----) * (-----); "DEGREES",
                       B1        D

                       B1       180
301  PRINT "B="; ATN (----) * (-----); "DEGREES",
                       A1        D

302  PRINT "C1="; SQR (A1↑2 + B1↑2)

305  GO TO 10
```

This much of the program would take care of solving Case I. We will now add lines to take care of Case V.

```
110  IF B1 <>0 THEN 400
          .
          .
          .
400  IF C1 <>0 THEN 500
          .
          .
          .
500 PRINT "A="; ATN(SQR((C1)↑2 - (B1)↑2)/B1)*180/D;"DEGREES",
501 PRINT "B="; ATN(B1/SQR((C1)↑ - (B1)↑2))*180/D; "DEGREES",
502 PRINT "A1="; SQR(C1↑2 - B1↑2)
505 GO TO 10
```

Complete the program so it will solve the remaining cases. Use your program to solve the following right triangles

(a) a = 150, B = 30°	(d) c = 23.5, a = 18.0
(b) b = 240, a = 250	(e) B = 70°, c = 55
(c) a = 192, A = 23°	(f) A = 37°, a = 267

Not all triangles encountered are right triangles. Suppose we were trying to determine the distance between points A and B in the following figure:

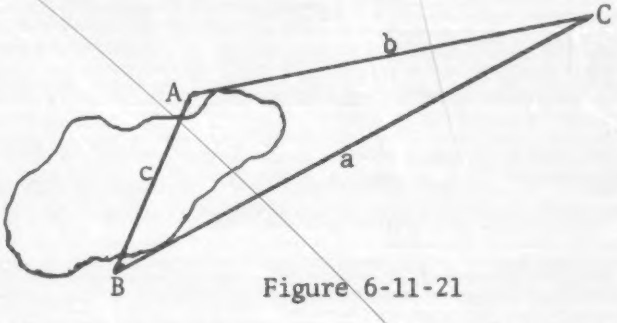


Figure 6-11-21

Assume it is impossible to walk from A to C at right angles to $\overline{AB}$. To be able to deal with situations like this we will develop two formulas that will make it possible to solve non-right triangles.

In the above Figure 6-11-21 we note that we could measure $\measuredangle C$, $\overline{AC}$ and $\overline{BC}$. We will now develop a formula for solving a triangle when we know the measure of two sides and the included angle.

In Section 6-10 we showed that the coordinates of any point P(x,y) in the Cartesian plane, r units from the origin, are $x = r \cos \theta$ and $y = r \sin \theta$. See figure 6-11-22 below.

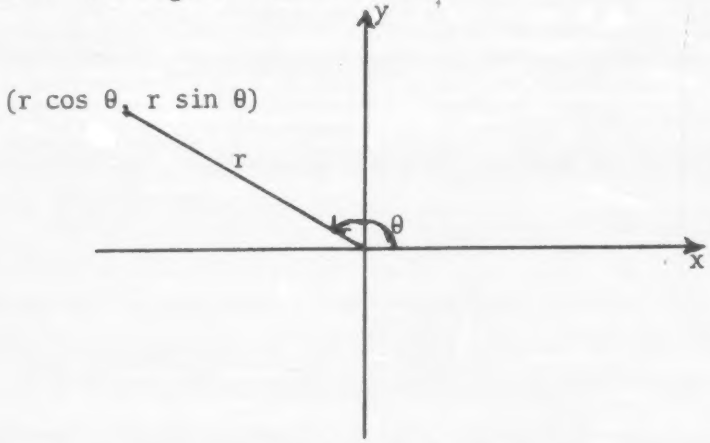


Figure 6-11-22

Consider an oblique triangle in the Cartesian plane with one vertex at the origin as illustrated below. (Figure 6-11-23)

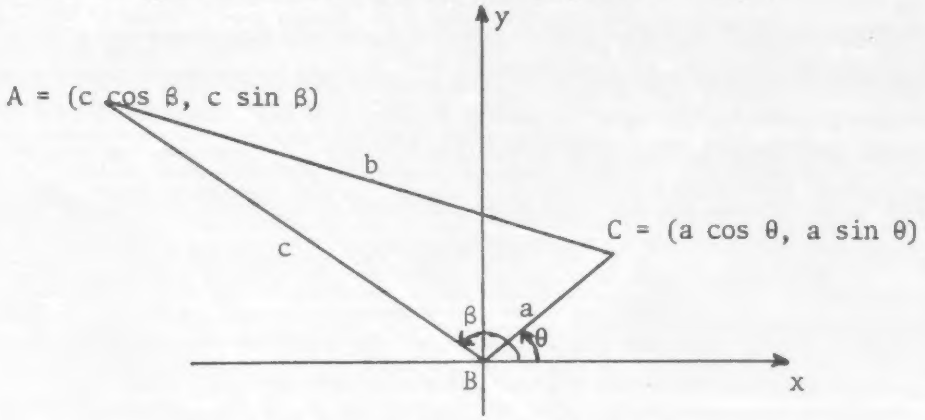


Figure 6-11-23

Do you see why the coordinates of A and C are as indicated? Using the distance formula we compute the measure of $\overline{AC}$.

$b^2 = (a \cos \theta - c \cos \beta)^2 + (a \sin \theta - c \sin \beta)^2$

$b^2 = a^2\cos^2\theta - 2ac \cos \theta \cos\beta + c^2\cos\beta^2 + a^2\sin^2\theta - 2ac \sin \beta \sin \theta + c^2\sin^2\beta$

$= a^2(\cos^2\theta + \sin^2\theta) + c^2(\cos^2\beta + \sin^2\beta) - 2ac(\cos \theta \cos\beta + \sin\beta \sin \theta)$

$= a^2 + c^2 - 2ac \cos (\beta - \theta)$

But $(\beta - \theta)$ is the measure of B therefore $b^2 = a^2 + c^2 - 2ac \cos B$. This final equation, along with two others derived by orienting the triangle with each of the other vertices at the origin, are known as:

The Law of Cosines:

(1) $a^2 = b^2 + c^2 - 2bc \cos A$

(2) $b^2 = a^2 + c^2 - 2ac \cos B$

(3) $c^2 = a^2 + b^2 - 2ab \cos C$

We now return to our problem, (See Figure 6-11-21) Assume we were able to measure C = 33°, $\overline{AC}$ = 90 yards, and $\overline{BC}$ = 125 yards. We wish to find c, the measure of $\overline{AB}$. We will use (3) of the law of cosines.

$c^2 = a^2 + b^2 - 2ab \cos C$

$c^2 = (125)^2 + 90^2 - 2(125)(90) \cos 33°$

$c^2 = 15,625 + 8100 - 225000(\cos 33°)$

$c^2 = 23,725 - 22500(.8387)$

$c^2 = 23725 - 18,871$

$c = \sqrt{4854} \approx 70$ yards

Below is a flow chart of solving a triangle by use of the law of cosines.

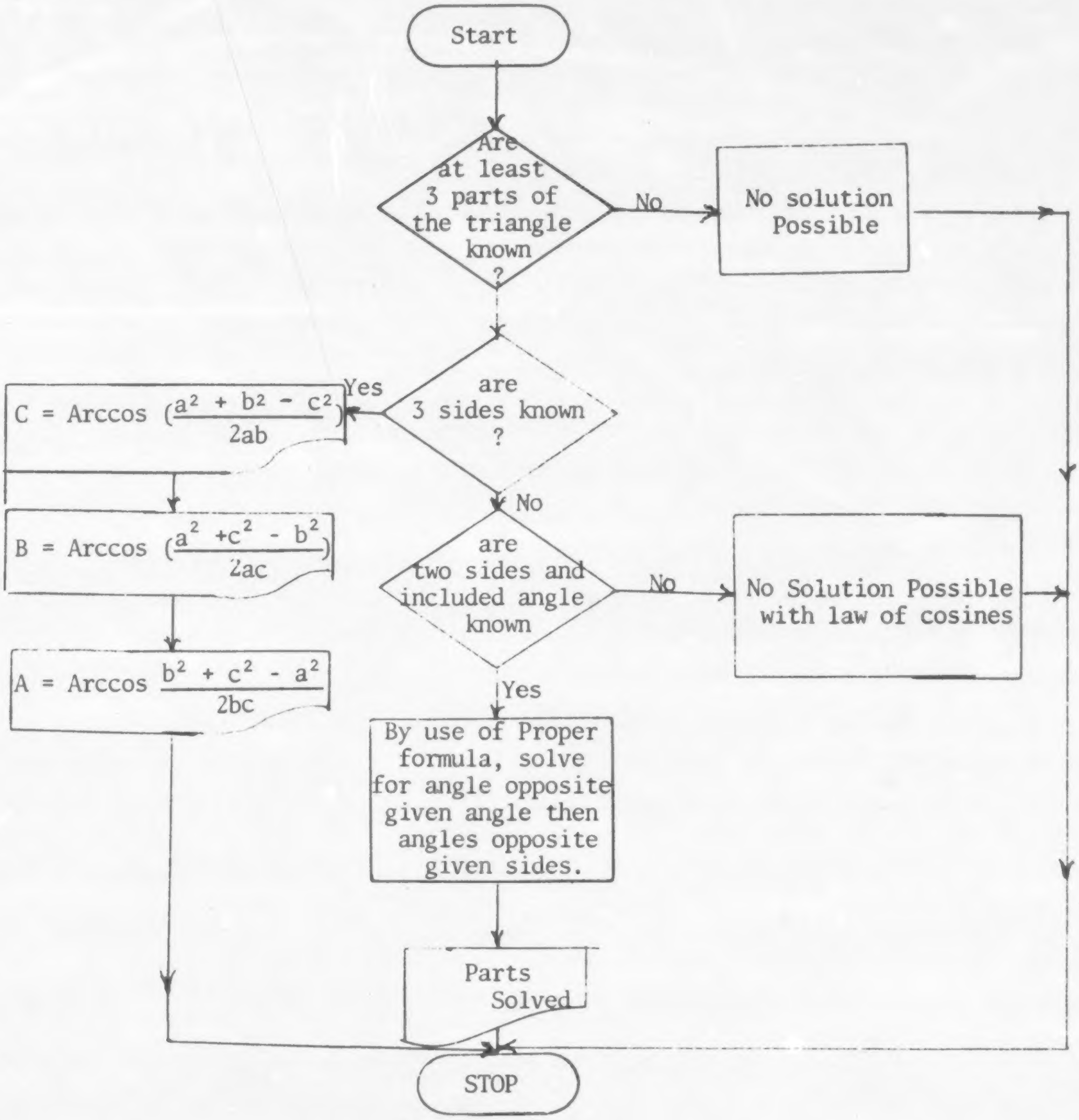

Start
Are at least 3 parts of the triangle known ?
No
No solution Possible
are 3 sides known ?
Yes
C = Arccos $(\frac{a^2 + b^2 - c^2}{2ab})$
B = Arccos $(\frac{a^2 + c^2 - b^2}{2ac})$
A = Arccos $\frac{b^2 + c^2 - a^2}{2bc}$
No
are two sides and included angle known
No
No Solution Possible with law of cosines
Yes
By use of Proper formula, solve for angle opposite given angle then angles opposite given sides.
Parts Solved
STOP

Figure 6-11-24

Exercise 6-11-25

1. Write a BASIC program from the above flow chart.
2. Use your program to solve the triangles with the following data:

(a) $a = 5$, $b = 6$, $c = 7$

(b) $a = 3$, $c = 4$, $b = 60°$

(c) $a = 8.17$, $b = 12.1$, $c = 15.21$

(d) $b = 2.71$, $c = 3.89$, $A = 24°30'$

(e) $a = 6.5$, $b = 8.7$, $C = 60°45'$

(f) $a = 8$, $b = 15$, $c = 17$

The laws of cosines will not solve all non-right triangles. For instance consider $\triangle ABC$ with $C = 33°$, $A = 127°$, $b = 90$ yards, find $\overline{AB}$. None of the formulas of the law of cosines can be used to find C in Figure 6-11-26.

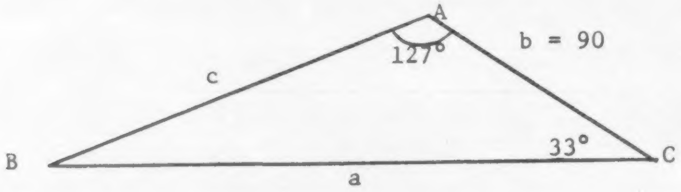


Figure 6-11-26

To be able to use the law of cosines we must know either the measures of two sides and the included angle or all three sides. Therefore, we will develop another series of formulas known as the law of sines. The law of sines, along with the law of cosines will be sufficient to solve any triangle, given three parts of the triangle.

Let $\triangle ABC$ be any triangle placed in the Cartesian plane with one vertex at the origin and the side opposite that vertex parallel to the x-axis as shown in the figure below.

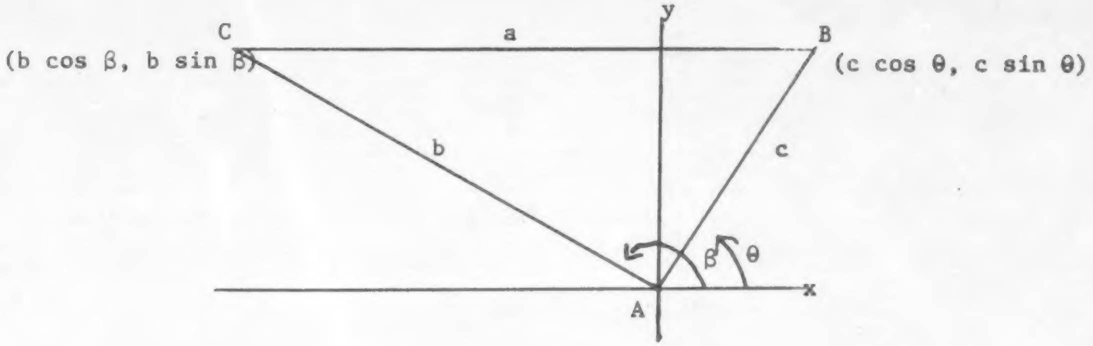


Figure 6-11-27

Since BC is parallel to the x-axis, $b \sin \beta = c \sin \theta$.

The measure of $\measuredangle B = \theta$ and the measure of $A + \measuredangle B = \beta$

therefore, $b \sin (A + B) = c \sin B$

$$A + B + C = 180° \rightarrow A + B = 180° - C$$

then $b \sin (180° - C) = c \sin B$

but $\sin (180° - C) = \sin C$

so $b \sin C = c \sin B$

or $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

By rotating the triangle in Figure 6-11-27 above, so that each of the other vertices is at the origin we can show that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ and $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$. Therefore we can state the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Now we can go back to the problem posed in Figure 6-11-26.

Given $C = 33°$, $A = 127°$, $b = 90$ yards, find c. Since $A + B + C = 180°$, $B = 180 - (A + C) = 20°$.

From the law of sines we have

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{90}{\sin 20^\circ} = \frac{c}{\sin 33^\circ}$$

$$c \sin 20^\circ = 90 \sin 33^\circ$$

$$c = 90 \cdot \frac{\sin 33^\circ}{\sin 20^\circ}$$

$$c = 90\ (.5446)/.3420$$

$$c = 143.31$$

Exercise 6-11-28

1. Make a flow chart of the process of solving a triangle by use of law sines.
2. Write a computer program to solve the problems which use the law of sines as described in the flow chart.
3. Solve the following triangles by use of your program.
   a. $A = 42°37'$, $B = 58°42'$ and $a = 24.64$
   b. $B = 53°17'$, $C = 42°46'$ and $c = 7.468$
   c. $C = 37°22'$, $a = 34.22$ and $b = 21.43$
   d. $A = 37°41'$, $b = 20.42$ and $a = 11.37$
   e. $A = 34°20'$, $a = 112.8$ and $c = 200$

The process of solving triangles is not always such a routine matter as might be inferred by the examples and exercises above. Decisions that have to be made in the process are:

1. Are three parts of the triangle known?
2. Do the parts given determine one, two or no triangles?
3. Which formula applies to the given triangle?

Question (1) above must be answered because both the law of sines and the law of cosines require that three parts be known in order to find a fourth part.

Question (2) needs to be answered because we remember from geometry that unique triangles are determined when we are given:

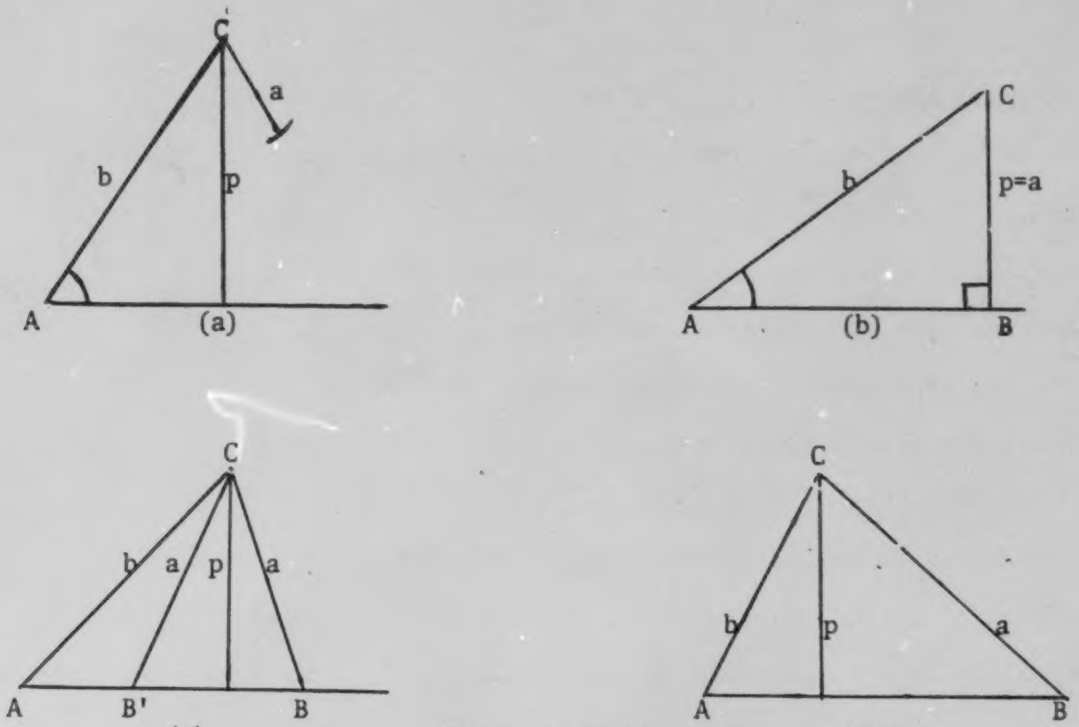
1. two angles and the side included between them (ASA)
2. two sides and the included angle (SAS)
3. three sides (SSS)
4. two angles and a side opposite one of them (AAS)

Therefore, if the data given corresponds to one of the above situations, there is a unique solution to the triangle.

Given the measure of the three angles of a triangle (AAA) there are an infinite number of solutions and such a problem would not have much practical significance.

Now comes the interesting situation. Given two sides and an angle opposite one of them (SSA) there are three possibilities: no solution, one solution or two solutions.

Let us examine the problem geometrically to see how the various situations arise. We first assume the given angle to be acute. Suppose the given parts are angle A, and sides a and b. Let us construct the given angle A and lay off the side b along one of its sides as indicated in Figure 6-11-29. The other extremity of b is the vertex C. Let p be the length of the perpendicular dropped from C to the other side of angle A.



(c) Figure 6-11-29 (d)

In each case $p = b \sin A$ and:

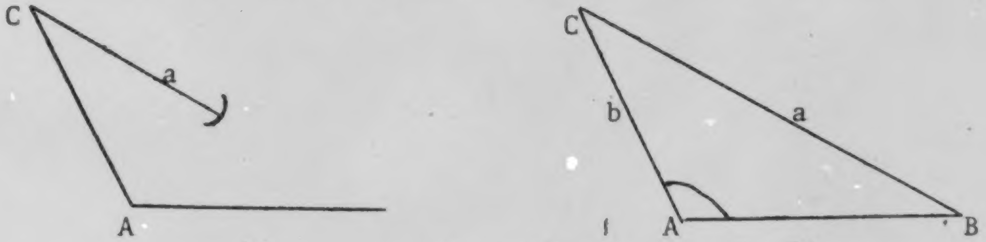
(1) $a < p$, in which case there is no triangle. (See Figure 6-11-29-a)

(2) $a = p$, in which case there is one triangle, which is a right triangle. (See Figure 6-11-29-b)

(3) $p < a < b$, in which case there are two triangles. (See Figure 6-11-29-c)

(4) $a \geq b$, in which case there is one triangle. (See Figure 6-11-29-d)

If the given angle A is obtuse, the student can easily convince himself from a study of Figure 6-11-30 that we have two cases.



(a) Figure 6-11-30 (b)

(1) $a \leq b$, in which case there is no triangle. (See Figure 6-11-30-a)

(2) $a > b$, in which case there is one triangle. (See Figure 6-11-30-b)

For convenience in our discussion above, we named the given parts A, a, and b. To make the statements concerning the conditions for the number of triangles applicable to every instance, regardless of what the given parts are called, we need merely to replace angle A by "given angle," side a by "opposite side," and side b by "adjacent side." Then, for the perpendicular p, we have

p = (adjacent side) x (sine of given angle.)

Statement (1) becomes "if the opposite side is less than p there is no triangle." The other statements (2), (3), and (4), which apply when the given angle is acute, can be translated into general terms in a similar manner. The student should write out the general statements at this point.

Exercise 6-11-31

1. Complete the following flow chart of the process of solving a triangle by use of the law of sines.

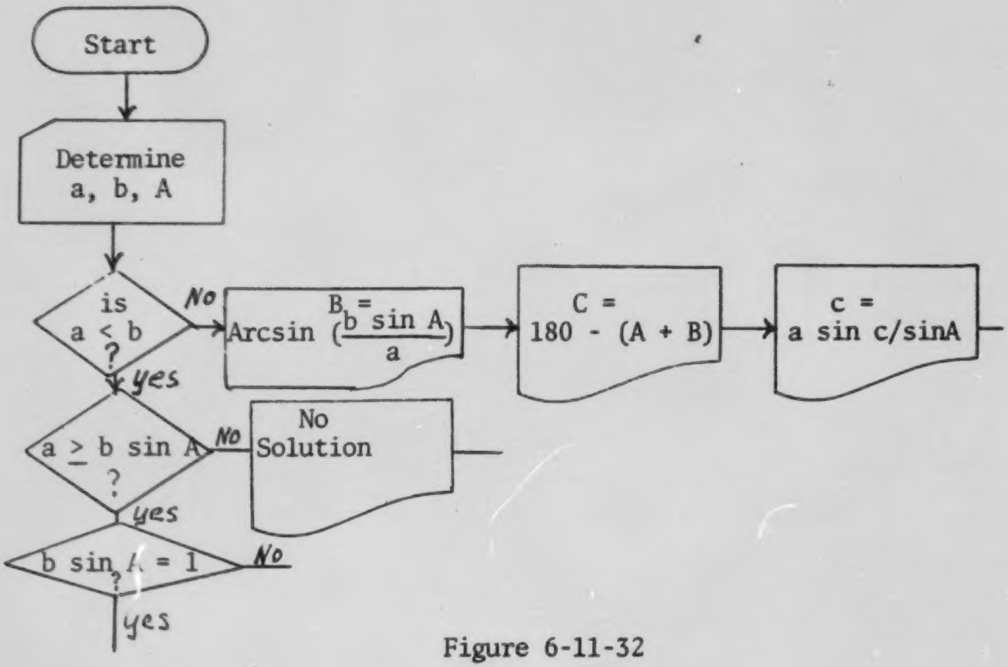


Figure 6-11-32

2. Write the program for your flow chart.

3. Solve the remaining parts of the triangle when the following measurements are given:

   (a) a = 11, b = 12, A = 63°

   (b) a = 17, b = 23, A = 68°

   (c) a = 7.5, b = 12.5, A = 48°35'

   (d) a = 8, b = 12, A = 23°40'

Below is a macro flow chart of the process of solving any triangle given any three parts.

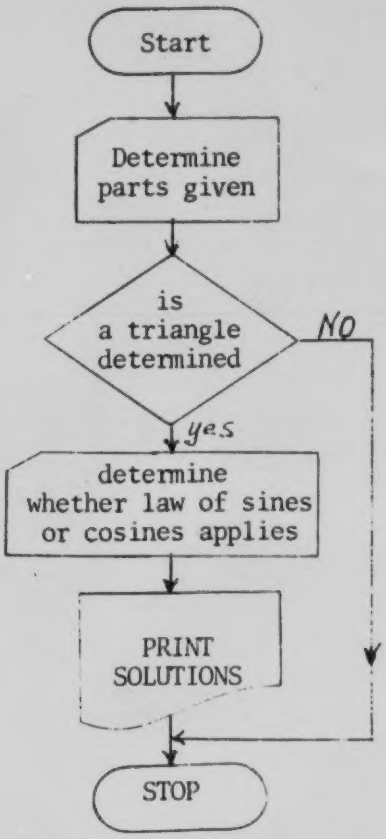


Figure 6-11-33

The first step in the above flow chart requires that one instruct the computer as to which parts of a particular triangle are given. In solving right triangles we used a coding method, which we could use again, but we will introduce another method which employs "string variables."

A string variable is a letter followed by a dollar sign, such as A$, M$, and so on. A string variable can be assigned to alphanumeric data (letter and numeral combinations) names, or other identifying information.

A simple program using a string variable follows:

```
10 READ M$, N$
15 FOR A = 1 TO 10
20 IF A/2 = INT(A/2) THEN 50
30 PRINT A; N$
40 GO TO 70
50 PRINT A; M$
60 DATA is EVEN, is ODD
70 NEXT A
80 END
```

The output of the above program would be:

```
 1 is ODD
 2 is EVEN
 3 is ODD
 4 is EVEN
     .
     .
     .
10 is EVEN
```

Let us see how we might write a program using string variables to solve any triangle given three parts. We will begin by using the most direct approach, that is, with an input statement.

```
10 INPUT G$
     .
     .
     .
     .
100 IF G$ = AAA THEN 200
110 IF G$ = ASA THEN 210
     .
     .
     .
     .
200 PRINT "NO TRIANGLE DETERMINED"
205 GO TO 10
210 INPUT A, C1, B
220 LET C = 180 - (A + B)
210 LET A1 = C1* sin A/ sin C
220 LET B1 = C1* sin B/ sin C
230 PRINT "ANGLE C ="; C, "SIDE A="; A1, "SIDE B="; B1
```

Exercise 6-11-34

1. Complete the above program.
2. Solve the following triangles, using the given information
   - (a) a = 236, b = 171, c = 217
   - (b) A = 52°, B = 43°, a = 617.2
   - (c) a = 31.2, b = 25.3, A = 36°
   - (d) b = 3.267, c = 3.013, C = 73.17°
   - (e) b = 1.1786, c = 1.3624, B = 56.1°
   - (f) b = .51623, c = .41678, A = 36.2
3. Find the information requested in each of the following situations.
   - (a) A tower stands on a hill which rises at an angle of 21° with the horizontal. From two points, A and B, on the slope and

in a line with the tower the angles of elevation of the top of the tower are 37° and 52°, respectively. If A is 180 ft. from B, find the height of the tower to the nearest foot.

(b) From a ship, two capes are sighted. One has bearing N35°W, and the other bears N20°W. The distance between the capes is 1.9 miles, and one cape has bearing N70°E from the other. How far from the cape is the ship?

(c) In order to measure the length of a proposed tunnel under a mountain, the distance from the entrance to a point P is 1200 yards and from the exit to point P is 1700 yards. If the angle subtended by the entrance and exit from P is 95°, how long is the proposed tunnel to be?

(d) A battery B is firing at an invisible target T and is directed by an observer at H that is 6186 yards, in the direction S26°31'W from B. If T is 1672 yards from H in the direction S27°43'E, find the distance and direction from B to T.

(e) The angle between the directions of two forces acting at a point is 61°12', and their resultant is 513.9 lbs. If one of the forces is 362.8 lbs., find the other.

(f) Find the radius of the inscribed circle and that of the circumscribed circle of a triangle of sides 70 ft., 82 ft., and 92 ft.

4. The formula, $K = \frac{1}{2}hb$, is a formula for finding the area of a triangle. Show that another formula for finding the area of a triangle is:

$$K = \frac{1}{2} bc \sin A$$

That is, the area of a triangle is equal to $\frac{1}{2}$ the product of any two sides and the sine of the included angle.

Use this formula to find the area of the triangle of Exercise 2(a), 2(b), and 2(c) above.

Values of Circular and Trigonometric Functions

Degrees	Real Number	Sine	Cosine	Tangent	Degrees	Real Number	Sine	Cosine	Tangent
0	.000	0.000	1.000	0.000					
1	.017	.018	1.000	.018	46	0.803	0.719	0.695	1.036
2	.035	.035	0.999	.035	47	.820	.731	.682	1.072
3	.052	.052	.999	.052	48	.838	.743	.669	1.111
4	.070	.070	.998	.070	49	.855	.755	.656	1.150
5	.087	.087	.996	.088	50	.873	.766	.643	1.192
6	.105	.105	.995	.105	51	.890	.777	.629	1.235
7	.122	.122	.993	.123	52	.908	.788	.616	1.280
8	.140	.139	.990	.141	53	.925	.799	.602	1.327
9	.157	.156	.988	.158	54	.942	.809	.588	1.376
10	.175	.174	.985	.176	55	.960	.819	.574	1.428
11	.192	.191	.982	.194	56	.977	.829	.559	1.483
12	.209	.208	.978	.213	57	.995	.839	.545	1.540
13	.227	.225	.974	.231	58	1.012	.848	.530	1.600
14	.244	.242	.970	.249	59	1.030	.857	.515	1.664
15	.262	.259	.966	.268	60	1.047	.866	.500	1.732
16	.279	.276	.961	.287	61	1.065	.875	.485	1.804
17	.297	.292	.956	.306	62	1.082	.883	.470	1.881
18	.314	.309	.951	.325	63	1.100	.891	.454	1.963
19	.332	.326	.946	.344	64	1.117	.899	.438	2.050
20	.349	.342	.940	.364	65	1.134	.906	.423	2.145
21	.367	.358	.934	.384	66	1.152	.914	.407	2.246
22	.384	.375	.927	.404	67	1.169	.921	.391	2.356
23	.401	.391	.921	.425	68	1.187	.927	.375	2.475
24	.419	.407	.914	.445	69	1.204	.934	.358	2.605
25	.436	.423	.906	.466	70	1.222	.940	.342	2.747
26	.454	.438	.899	.488	71	1.239	.946	.326	2.904
27	.471	.454	.891	.510	72	1.257	.951	.309	3.078
28	.489	.470	.883	.532	73	1.274	.956	.292	3.271
29	.506	.485	.875	.554	74	1.292	.961	.276	3.487
30	.524	.500	.866	.577	75	1.309	.966	.259	3.732
31	.541	.515	.857	.601	76	1.326	.970	.242	4.011
32	.559	.530	.848	.625	77	1.344	.974	.225	4.331
33	.576	.545	.839	.649	78	1.361	.978	.208	4.705
34	.593	.559	.829	.675	79	1.379	.982	.191	5.145
35	.611	.574	.819	.700	80	1.396	.985	.174	5.671
36	.628	.588	.809	.727	81	1.414	.988	.156	6.314
37	.646	.602	.799	.754	82	1.431	.990	.139	7.115
38	.663	.616	.788	.781	83	1.449	.993	.122	8.144
39	.681	.629	.777	.810	84	1.466	.995	.105	9.514
40	.698	.643	.766	.839	85	1.484	.996	.087	11.43
41	.716	.658	.755	.869	86	1.501	.998	.070	14.30
42	.733	.669	.743	.900	87	1.518	.999	.052	19.08
43	.751	.682	.731	.933	88	1.536	.999	.035	28.64
44	.768	.695	.719	.966	89	1.553	1.000	.018	57.29
45	.785	.707	.707	1.000	90	1.571	1.000	.000	undefined

Chapter 7

Quadratic Functions

7-1 INTRODUCTION

To introduce quadratic functions we will consider the behavior of a ball that is dropped or thrown into the air from the top of a tall building.

Physicists have shown that in a vacuum, a falling object travels $(1/2)gt^2$ feet during t seconds where g is the gravitational acceleration approximately equal to 32.16 ft/sec.^{2}. To simplify our discussion we will use $g = 32$ ft/sec^2. Hence, we can say that a falling object will travel $1/2(32)t^2$ or $16t^2$ feet in t seconds.

We now want to develop a mathematical description of the ball's motion as it travels up or down from the top of the building. In order to describe the displacement (location) of the ball at any time t, we must place an imaginary number line against the side of the building. Such a scale is shown in Figure 7-1-1. Zero is located at the top of the building. Upward displacements of the ball are represented by the positive half of the number line and downward displacements by the negative half.

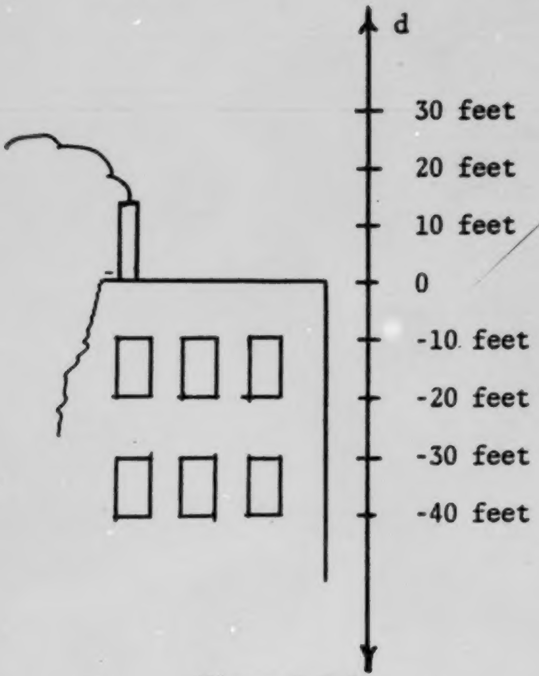


Figure 7-1-1

Now assume the ball is dropped straight down from the top of the building. Remember the distance traveled is $16t^2$ feet in t seconds and the ball is moving downward (in a negative direction). We can see that the displacement of the ball would be given by the equation:

$$d = -16t^2$$

For example, after one second at $t = 1$ the displacement would be

$$d = -16(1)^2$$

$$= -16 \text{ ft.}$$

The ball would have a displacement of -16 feet from the roof top when $t = 1$ second.

Exercise 7-1-2

1. What is the displacement of the ball at time:

    a. $t = 2$ seconds

    b. $t = 3$ seconds

    c. $t = 4$ seconds

    d. $t = 5$ seconds

    e. $t = 6$ seconds

2. Plot a graph of displacement versus time on the interval $0 \leq t \leq 6$.

3. Assume that the building is 576 feet tall. The displacement-time relation can be described by:

$$d = \{(t,d(t)) \mid d(t) = -16t^2,\ 0 \leq t \leq 6\}$$

(a) What is the physical significance of the restriction $0 \leq t \leq 6$.

(b) Is the relation d a function?

Now consider a variation in the situation. The ball is thrown straight up (in the positive direction) from the roof top with an initial velocity of 80 ft/sec. The displacement of the ball is now given by the equation:

$$d = -16t^2 + 80t$$

For example at $t = 0$ the ball would have a displacement of

$$d = -16(0)^2 + 80(0)$$

$$d = 0$$

because it has not yet been thrown when $t = 0$.

However, after 1 second at $t = 1$ the displacement would be

$$d = -16(1)^2 + 80(1)$$

$$= +64$$

The ball would have a displacement of +64 feet from the roof top when $t = 1$.

Exercise 7-1-3

1. What is the displacement of the ball at the following times?

   $(d = -16t^2 + 80t)$

   (a) $t = 2$ seconds

   (b) $t = 3$ seconds

   (c) $t = 4$ seconds

   (d) $t = 5$ seconds

   (e) $t = 6$ seconds

2. Plot a graph of d versus t for this new situation.

3. How can the ball have a zero displacement after 5 seconds at $t = 5$ as shown in the graph?

4. If the building is 576 feet tall how many seconds elapse before the ball strikes the ground?

5. What value of n should be used in the following description of this new displacement-time relation?

   $$d = \{(t,d(t)) | d(t) = -16t^2 + 80t,\ 0 \leq t \leq n\}$$

6. Is this displacement-time relation a function?

We will consider one more variation in the situation. Suppose the ball was thrown straight up from an initial displacement of -96 feet with an initial velocity of 80 ft/second. The displacement of the ball in this new situation is given by the equation.

$$d = -16t^2 + 80t - 96$$

For example, after 1 second at t = 1 sec. the displacement would be

$$d = -16(1)^2 + 80(1) - 96$$
$$= -32 \text{ ft.}$$

The ball would have a displacement of -32 feet from the roof top when t = 1 second.

Exercise 7-1-4

1. Plot a graph of this new displacement time relation.

$$d = \{(t,d(t)) | d(t) = -16t^2 + 80t - 96\}$$

2. How many seconds have elapsed before the ball passes the roof top?

   (a) On the way up?

   (b) On the way down?

Careful analysis of these three displacement-time relations reveals that all of them are functions. In addition we can see that each of the "mathematical models" or "set selectors" is an equation of the form

$$d = at^2 + bt + c; \ a, b, c \in R; \ a \neq 0$$

These three examples are illustrative of an important class of functions called quadratic functions because they satisfy the following definition.

Definition 7-1-5 Quadratic Function

Any function of the form

$$\{(x,y) | y = ax^2 + bx + c, \ a, b, c \in R, \ a \neq 0\}$$ is a

quadratic function.

Example 7-1-6

The function $\{(t,d(t)) | d(t) = -16t^2\}$ is a quadratic function where

$a = -16$, $b = 0$ and $c = 0$.

Example 7-1-7

The function $\{(x,y) | y = 2x + 1\}$ is NOT a quadratic function because $a = 0$.

Exercise 7-1-8

Which of the following equations define a quadratic function?

1. $y = x^2$
2. $y = 2x$
3. $y = 2x^2$
4. $y = \frac{2}{x^2}$
5. $y = x^2 + 1$
6. $y = 2x + 1$
7. $y = x^2 + x$
8. $y = x(x - 1)$
9. $y = x(x - 1)(x - 2)$
10. $y = (2x)^2$

For what values of t do the following equations define a quadratic function?

11. $y = tx^2 + 3x + 4$
12. $y = (t - 2)x^2 + 1$
13. $y = x^t + 2x + 3$

Each of the following equations is equivalent to an equation of the form $y = ax^2 + bx + c$. For each find a, b, and c.

14. $y = 3(x - 4)^2$
15. $y = (x + 2)(x - 3)$

7-2 Quadratic Function

We are going to study quadratic functions by examining a succession of special cases. We begin with the function defined by $y = x^2$, and then progress to functions defined by equations in each of the following forms:

(1) $y = ax^2$

(2) $y = ax^2 + c$

(3) $y = a(x - k)^2$

(4) $y = a(x - k)^2 + p$

Eventually, we will arrive at the general case of functions defined by equations of the form

$$y = ax^2 + bx + c$$

In each case we shall try to see what the graphs of these functions look like and how changing the values of a, b, and c affects these graphs.

7-3 The Equation $y = x^2$

The equation $y = x^2$ defines a function f whose domain is the set of real numbers. Its range is the set of all non-negative real numbers. For any non-zero number and its opposite, both elements of the domain, the resulting element of the range is the same. This is known to be true because $\forall a$, $(a)^2 = (-a)^2$. If 0 is the element of the domain being considered then the element of the range that it maps into is also 0. Observe this in the example below.

Example 7-3-1

Plot the graph of $f = \{(x,y) \mid y = x^2\}$

Solution: List a table of values:

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Plot these points and draw a smooth curve through them:

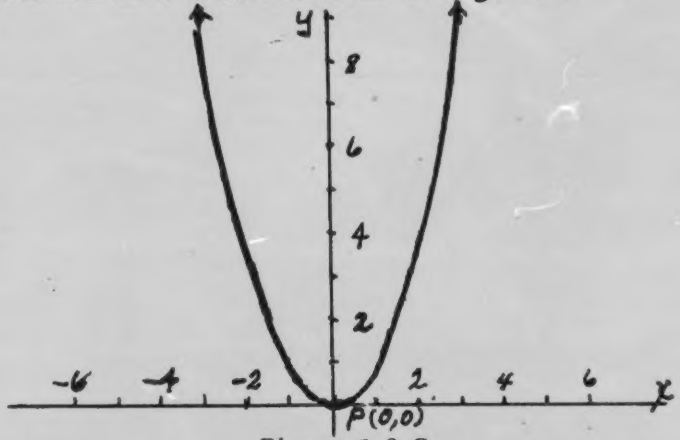


Figure 7-3-2

The curve of Figure 7-3-2 is called a parabola, the point p is called its vertex, or critical point, and the line x = 0 is called the axis of symmetry.

Exercise 7-3-3

1. Discuss the differences in the graphs of the following functions:

$f = \{(x,y) | y = x^2, x \epsilon R\}$; $g = \{(x,y) | y = x^2, x \epsilon I\}$:

$h = \{(x,y) | y = x^2, x \epsilon Q\}$

7-4 Equations of the Form $y = ax^2$

For each value of a, the equation $y = ax^2$ defines a function. These functions are best studied in two cases:

Case I: $a > 0$

(1) For $y < 0$, there are no values of x which satisfy $y = ax^2$.

(2) For $y = 0$, there is one value of x which satisfies $y = ax^2$, namely 0.

(3) For each $y > 0$, there are two values of x which satisfy $y = ax^2$, namely $\sqrt{\frac{y}{a}}$ and $-\sqrt{\frac{y}{a}}$.

Exercise 7-4-1

1. On the same coordinate axis graph the relations defined by $y = ax^2$ when $a = 1/10, 1/2, 1, 2, 5$.

2. You should notice that the smaller values of $|a|$ correspond to the wider curve in the graphs of Problem 1.

   (a) Which way does each graph open, upward or downward?

   (b) What is the axis of symmetry of each function?

Case II: $a < 0$

The graph of $y = ax^2$, $a < 0$, can be sketched easily from the graph of $y = -ax^2$. One will be the x-axis mirror image of the other.

Example 7-4-2

Plot a graph of $y = -4x^2$

The graph of this equation is the x-axis mirror image of the graph of $y = 4x^2$ as shown in the Figure below.

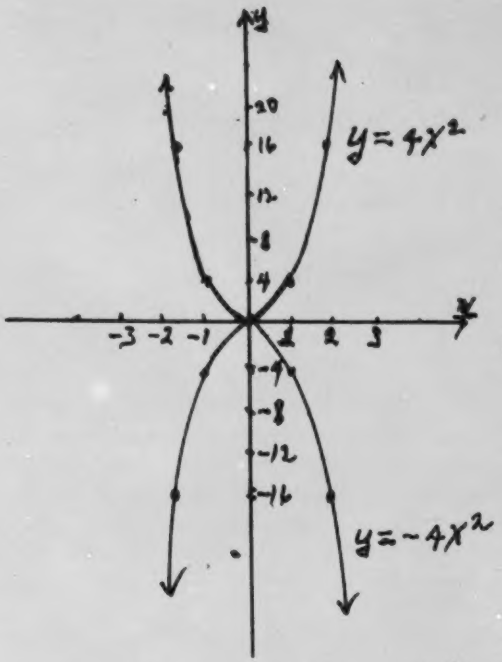


Figure 7-4-3

Note, that if a mirror were placed on the y-axis the graph of $y = -4x^2$ would be a reflection of the graph of $y = 4x^2$. If the paper were folded along the x-axis, the graph of $y = 4x^2$ would be superimposed on the graph of $y = -4x^2$.

We can generalize by saying that graphs of functions defined by equations of the form $y = ax^2$ will open upward if $a > 0$ and will open downward if $a < 0$. Hence, for each real number a, the graphs of $y = ax^2$ and $y = -ax^2$ will be x-axis mirror images.

Exercise 7-4-4

1. (a) On the same coordinate axis used in Exercise 7-4-1 sketch the graphs of the functions defined by $y = ax^2$, where $a = -1/10, -1/2, -1, -2, -5$.

   (b) What are the x-axis mirror images of these graphs?

   (c) Do the functions have maximum or minimum values.

2. Plot the graph of each of the following equations:

   (a) $y = 3x^2$ (c) $y = 1/3x^2$

   (b) $y = -3x^2$ (d) $y = -1/3x^2$

3. For each of the following determine a, so that the graph of $y = ax^2$ contains the given point:

   (a) (1,2)

   (b) (1,-1)

   (c) (-2,2)

7-5 Equations of the Form $y = ax^2 + c$

Let us now consider the graphs of functions defined by equations of the form $y = ax^2 + c$. The Figure 7-5-1shows the graphs of four functions which are representative of this case. The functions are defined by:

(1) $y = x^2 + 1$

(3) $y = x^2 - 1$

(2) $y = x^2 + 2$

(4) $y = x^2 - 2$

So that you may compare these graphs with that of the familiar $y = x^2$, the graph of the latter has been sketched in with a dashed line.

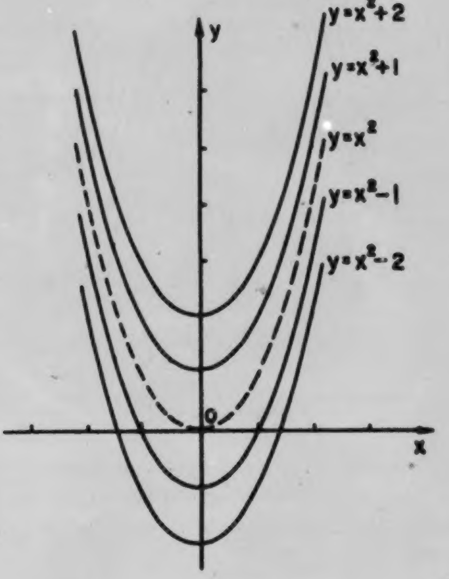


Figure 7-5-1

By studying Figure 7-5-1 you can see that for any x the ordinate of $y = x^2 + 2$ is two units greater than the corresponding ordinate of $y = x^2$. Similarly for any x the ordinate of $y = x^2 - 2$ is two units less than the corresponding ordinate of $y = x^2$. Thus, the lowest point on the graph of $y = x^2 + 2$ is $(0,2)$ and the lowest point on the graph of $y = x^2 - 2$ is $(0,-2)$. Note that each of these graphs has a minimum point.

Figure 7-5-2 shows the graph of $y = -x^2 + p$ for various values of p. Notice that in this case each of these graphs has a maximum point and opens downward.

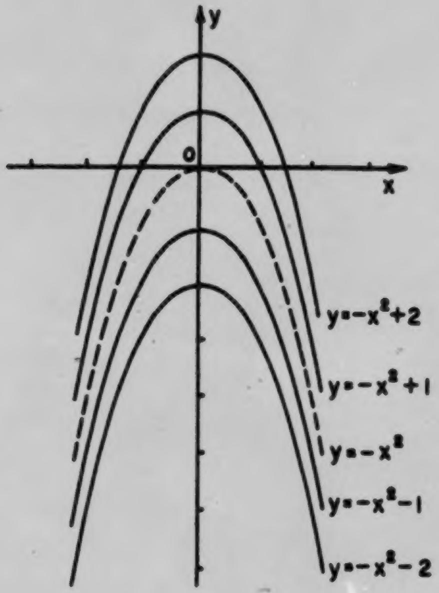


Figure 7-5-2

From our previous work with symmetry we see that functions defined by equations of the form $y = ax^2 + p$ are symmetric with respect to the y-axis regardless of the particular values of a and p. As before, the larger values of $|a|$ give "narrower" curves.

We may summarize by saying that for each real number a, the graph of $y = ax^2 + p$ is congruent to the graph of $y = ax^2$. However, it has a position which is p units above or below the x-axis according as p is positive or negative. In each case the curve cuts the y-axis at $(0,p)$.

Exercise 7-5-3

1. Find the critical point (highest or lowest) of the graph of each of the following equations and tell whether it is a maximum or minimum value.

   (a) $y = 5x^2 + 1$

   (b) $y = -5x^2 + 2$

2. Sketch the graph of each equation in Problem 1.

7-6 Equations of the Form $y = a(x - k)^2$

In this section we study functions defined by equations of the form

$$y = a(x - k)^2$$

where a and k are non-zero constants. We proceed by considering several examples.

Example 7-6-1

Make a table of values and plot the graph of $y = 2(x - 3)^2$

x	...	1	2	3	4	5
$y = 2(x - 3)^2$	...	8	2	0	2	8

The axis of symmetry of this curve is the line $x = 3$. Its critical point is the point (3,0).

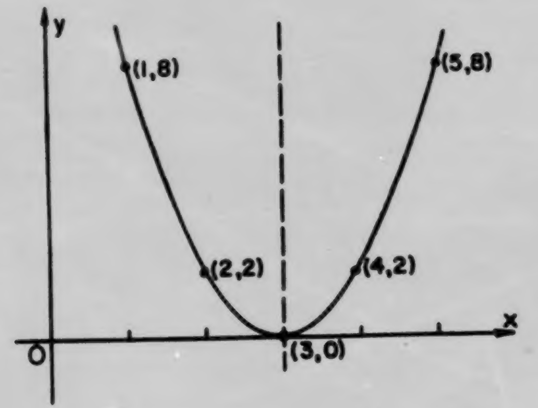


Figure 7-6-2

Example 7-6-3

Make a composite table of values for $y = 2x^2$ and $y = 2(x - 3)^2$ and plot the graphs of the two functions on the same set of axes.

x	···	-2	-1	0	1	2	3	4	5	···
$y = 2x^2$	···	8	2	0	2	8	18	···	···	···
$y = 2(x - 3)^2$	···	···	···	18	8	2	0	2	8	···

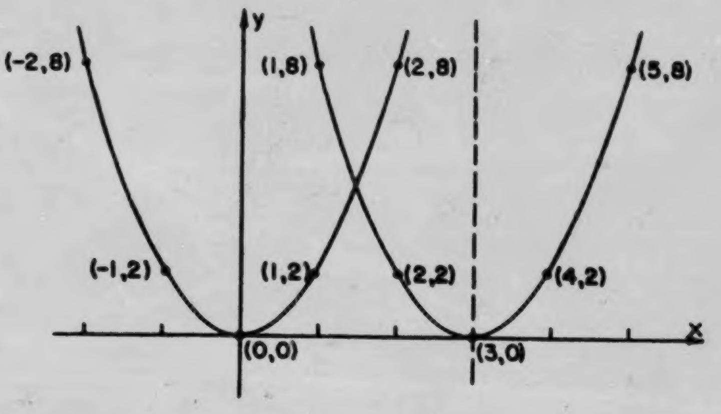


Figure 7-6-4

The graph of $y = 2x^2$ is symmetric with respect to the line $x = 0$, and the graph of $y = 2(x - 3)^2$ is symmetric with respect to the line $x = 3$.

Summary for Equations of the Form $y = a(x - k)^2$

1. The graph is congruent to the graph of $y = ax^2$.
2. It has a position $|k|$ units to the right or left of the graph $y = ax^2$ according as $k > 0$ or $k < 0$.
3. If $a > 0$, the graph opens upward and has a lowest point $(k,0)$; if $a < 0$ the graph opens downward and has a highest point $(k,0)$.
4. The graph is symmetric with respect to the line $x = k$, and this line is called the axis of symmetry.

Exercise.7-6-5

1. Find the critical point and the axis of symmetry of the graph of each of the following equations:

   (a) $y = (x - 2)^2$

   (b) $y = -2(x + 1)^2$

   (c) $y = -3(x + 0)^2$

2. Which of the equations of Problem 1 have graphs which open upward?
3. Sketch the graph of each of the equations in Problem 1.

7-7 Equations of the form $y = a(x - k)^2 + p$

We know that the graph of $y = ax^2 + p$ has a position which is $|p|$ units up or down from the graph of $y = ax^2$, and from the last section we know that the graph of $y = a(x - k)^2$ has a position that is $|k|$ units to the right or left of the graph of $y = ax^2$. Hence, the graph of $y = a(x - k)^2 + p$ is congruent to the graph of $y = ax^2$ but is $|p|$ units up or down and $|k|$ units to the right or left of the graph of $y = ax^2$. The expressions "up" and "to the right" are associated with positive values of p and k, and "down" and "to the left" are associated with negative values.

Example 7-7-1

Plot the graphs of $y = 2(x + 3)^2 + 1$, and $y = 2(x - 3)^2 + 1$ using a single set of axes.

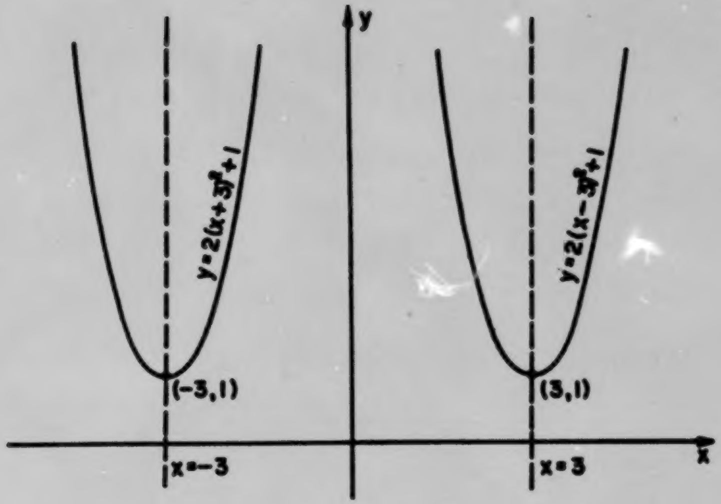


Figure 7-7-2

The graph of $y = 2(x - 3)^2 + 1$ has a lowest point (3,1) and has the line $x = 3$ as its axis of symmetry. The graph of $y = 2(x + 3)^2 + 1$ has a lowest point (-3,1) and has the line $x = -3$ as its axis of symmetry. Notice that both curves open upward.

Summary for equations of the form $y = a(x - k)^2 + p$

1. If $a > 0$ the graph opens upward and the curve has a lowest point (k,p). If $a < 0$ the graph opens downward and has a highest point (k,p).

2. The graph has the line $x = k$ as its axis of symmetry.

Exercise 7-7-3

1. Find the vertex and the axis of symmetry of the graph of each of the following equations:

   (a) $y = 2(x - 3)^2 + 4$
   (b) $y = -2(x - 3)^2 + 4$
   (c) $y = (x + 3)^2$
   (d) $y = -1/2(x - 1)^2 - 1$
   (e) $y = 3(x + 1)^2 + 2$
   (f) $y = 1/5(x - 2)^2 - 3$

2. Sketch the graph of each of the equations in Problem 1.

3. Which graphs of the exercises in Problem 1 have a minimum value and which have a maximum value? What are these values?

7-8 Equations of the Form $y = ax^2 + bx + c$

We turn now to general quadratic functions defined by equations of the form $y = ax^2 + bx + c$, and reduce the study of these functions to the special cases studied in the previous sections. We do this by performing a useful algebraic manipulation known as "completing the square."

First, we will consider a group of quadratic expressions known as perfect squares. Consider the quadratic expression $x^2 + 2x + 1$. By the distributive principle of the real numbers, we can show that $x^2 + 2x + 1 = (x + 1)(x + 1)$.

$$= (x + 1)^2$$

Likewise $$x^2 - 4x + 4 = (x-2)(x - 2)$$

$$= (x - 2)^2$$

and $$x^2 + 2\sqrt{5}x + 5 = (x + \sqrt{5})(x + \sqrt{5})$$

$$= (x + \sqrt{5})^2$$

Quadratic expressions, such as these, that can be expressed as linear expressions squared, are referred to as "perfect squares."

Exercise 7-8-1

1. Express each of the following quadratic expressions as a linear expression squared.

   (a) $x^2 - 6x + 9$

   (b) $x^2 + 10x + 25$

   (c) $x^2 + 8x + 16$

   (d) $x^2 - x + 1/4$

   (e) $x^2 + 2/3x + 1/9$

2. Supply the missing term so that the expression is a perfect square:

   (a) $x^2 - 12x + ?$

   (b) $x^2 - ? + 1/16$

   (c) $x^2 + 5x + ?$

   (d) $x^2 + ? + 9/4$

   (e) $x^2 + 6 + 2x + ?$

   (f) $x^2 - ? + 5/9$

You should recognize from the above exercises that if a quadratic expression of the form $x^2 + bx + c$, is a perfect square, then $c = (b/2)^2$. Also, $x^2 + bx + c = (x - k)^2$, where $k = \sqrt{c}$ if $b > 0$ and $k = -\sqrt{c}$ if $b < 0$.

There is another group of quadratic expressions which are classified as "perfect squares." For example,

$$\begin{aligned} 4x^2 + 20x + 25 &= 4(x^2 + 5x + 25/4) \\ &= 4(x + 5/2)^2 \\ &= (2(x + 5/2))^2 \\ &= (2x + 5)^2 \end{aligned}$$

Since $(2x + 5)^2$ is a linear expression squared we can classify $4x^2 + 20x + 25$ as a perfect square quadratic expression.

Example 7-8-2 Write each of the expressions $x^2 - 6x + 9$ and $x^2 + 6x + 9$ as a linear expression squared.

$x^2 - 6x + 9 = (x - \sqrt{9})^2 = (x + -6/2)^2$ while

$x^2 + 6x + 9 = (x + \sqrt{9})^2 = (x + 6/2)^2$

Exercise 7-8-3

Express each of the following as a linear expression squared:

(a) $9x^2 - 6x + 1$

(b) $16x^2 + 16x + 4$

(c) $25x^2 - 30x + 9$

(d) $4x^2 + 12x + 9$

Many quadratic expressions are not perfect squares. For example, if we were asked to express $2x^2 + 12x + 20$ as a linear expression squared, we would begin as before and write $2x^2 + 12x + 20 = 2(x^2 + 6x + 10)$. Now $x^2 + 6x + 10$ is not a perfect square because $(6/2)^2 \neq 10$. If the last term, 10, were a 9 it would be a perfect square. So we write $x^2 + 6x + 10 = x^2 + 6x + 9 + 1$. Then

$$\begin{aligned} 2x^2 + 12x + 20 &= 2(x^2 + 6x + 9 + 1) \\ &= 2(x^2 + 6x + 9) + 2 \\ &= 2(x + 3)^2 + 2 \end{aligned}$$

We were not able to express $2x^2 + 12x + 20$ as a linear expression squared, but we were able to express it in the form $a(x - k)^2 + p$. The manipulation we went through to arrive at the $a(x - k)^2 + p$ form is called "completing the square."

Example 7-8-4

Complete the square on $x^2 - 8x + 14$

$$x^2 - 8x + 14 = x^2 - 8x + (8/2)^2 - 2$$

$$= (x - 4)^2 - 2$$

This is of the form $a(x - k)^2 + p$, therefore we have completed the square.

Example 7-8-5

$$3x^2 + 6x + 5 = 3(x^2 + 2x + 5/3)$$

$$= 3(x^2 + 2x + 1 + 2/3)$$

$$= 3(x^2 + 2x + 1) + 3(2/3)$$

$$= 3(x + 1)^2 + 2$$

This is of the form $a(x - k)^2 + p$.

Exercise 7-8-6

1. By completing the square transform the following equations to the form $y = a(x - k)^2 + p$.

   (a) $y = x^2 - 4x$

   (b) $y = 3x^2 - 18x + 27$

   (c) $y = x^2 + 4x + 6$

   (d) $y = x^2 + 2x - 3$

   (e) $y = 3x^2 + 12x + 5$

   (f) $y = 2x^2 + 20x + 24$

   (g) $y = 10 + 5x - 5x^2$

2. Write a computer program that would print a quadratic expression $ax^2 + bx + c$ in the form $a(x - k)^2 + p$.

We shall now show that any expression of the form $ax^2 + bx + c$, $a \neq 0$, can be written in the form $a(x - k)^2 + p$.

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x + c/a)$$

$$ax^2 + bx + c = a(x^2 + \frac{b}{a} x + (b/2a)^2 + c/a - (b/2a)^2)$$

$$ax^2 + bx + c = a(x^2 + \frac{b}{a} x + (b/2a)^2) + a \cdot c/a - a(b/2a)^2$$

$$ax^2 + bx + c = a(x + b/2a)^2 + c - b^2/4a$$

$$ax^2 + bx + c = a(x + b/2a)^2 + \frac{4ac - b^2}{4a}$$

The expression $ax^2 + bx + c$ has been transformed into the general form $a(x - k)^2 + p$ with $k = -\frac{b}{2a}$ and $p = \frac{4ac - b^2}{4a}$

From the above we can conclude that:

$$\{(x,f(x)|f(x)= ax^2 + bx + c\}= \{(x,f(x)|f(x) = a(x + \frac{b}{2a})^2 + \frac{4ac - b}{4a}\}$$

The completed square form above gives us another approach to determine the critical point of a quadratic function. Examine the equation:

$$f(x) = a(x + b/2a)^2 + \frac{4ac - b^2}{4a}$$

Note that the term $\frac{4ac - b^2}{4a}$ maintains a constant value for all real number replacements of x, while the term $a(x + b/2a)^2$ varies as x varies. If $a > 0$ then the graph of the function opens upward, i.e., the function has a minimum value. Also, note that if $a > 0$, $a(x + b/2a)^2 \geq 0$ for all x, (why?). The value, f(x) of the function will be minimum when the sum of the terms on the right side of the equation $f(x) = a(x + b/2a)^2 + \frac{4ac - b^2}{4a}$ is minimum. Since

$\frac{4ac - b^2}{4a}$ is constant, the minimum value of the function is obtained when $a(x + b/2a)^2$ is minimum. We established above that $a(x + b/2a)^2 \geq 0$ for all x, therefore, the minimum value of this term is 0. When does $a(x + b/2a)^2 = 0$? Obviously when $x = \frac{-b}{2a}$. So we conclude f(x) is minimum when $x = \frac{-b}{2a}$.

Now we ask what is this minimum value?

If $x = \frac{-b}{2a}$ then

$$\begin{aligned} f(x) &= f(\tfrac{-b}{2a}) \\ &= a(\frac{-b}{2a} + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a} \\ &= \frac{4ac - b^2}{4a} \end{aligned}$$

We see the minimum value of the function is $\frac{4ac - b^2}{4ac}$ and the critical point f(x,y) of the graph is $P(-\frac{b}{2a}, \frac{4ac - b^2}{4a})$.

Example 7-8-7

What is the critical point of the following function?

$$f = \{(x,f(x)|f(x) = 2x^2 - 3x - 2\}$$

Solution:

$$a = 2,\ b = -3, \text{ and } c = -2$$

$$x = -\frac{b}{2a} = -\frac{-3}{2(2)} = \frac{3}{4}$$

$$f(x) = \frac{4ac - b^2}{4a} = \frac{4(2)(-2) - (-3)^2}{4(2)} = \frac{-16 - 9}{8} = -\frac{25}{8}$$

Therefore the critical point is $(3/4, -\frac{25}{8})$. The graph of f is shown in Figure 7-8-8

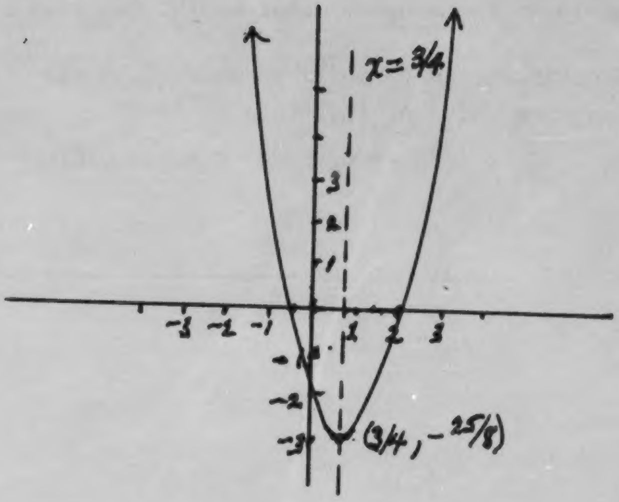


Figure 7-8-8

Now consider the original equation $f(x)=a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$ when $a < 0$. The graph of $f(x) = a(x + \frac{b}{2a}) + \frac{4ac - b^2}{4a}$ opens downward and has a maximum value. The term $a(x + \frac{b}{2a})^2 < 0$ for all replacements of x. (Why?). Therefore the maximum value of the expression $a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$ occurs when $x + \frac{b}{2a} = 0$, that is when $x = -\frac{b}{2a}$. The ordinate of the critical point is again $\frac{4ac - b^2}{2b}$.

So we summarize that the function f, defined by $f(x) = ax^2 + bx + c$, $a \neq 0$;

(1) has as its critical point the point $(-\frac{b}{2a}, \frac{4ac - b^2}{4a})$

(2) $x = -\frac{b}{2a}$ is the equation of the axis of symmetry of f.

(3) $\frac{4ac - b^2}{4a}$ is the maximum value of f if $a < 0$ and the minimum value of f if $a > 0$.

Exercise 7-8-9

1. For each of the following functions:

   (a) Find the coordinates of the critical point.

   (b) Write the equation of the axis of symmetry of the graph.

   (c) Indicate whether the graph opens upward or downward.

   (d) What is the maximum or minimum value of each function?

(e) Graph each function:

(1) $y = x^2 + 7x - 8$

(2) $y = -x^2 - 11x - 31$

(3) $y = -2x^2 - x - 1$

(4) $y = 4x^2 + x - 3$

(5) $y = -2x^2 - 5x - 1$

(6) $y = x^2 - \frac{7}{2}x + 3$

(7) $y = 5x^2 + 4x + 3$

(8) $y = -3x^2 + 2x - 2$

(9) $y = -5x^2 + 3x$

(10) $y = 2x^2 + 8$

2. Write a computer program which will perform the instructions (a) through (d) of problem 1. Test your program on the equations of problem 1.

7.9 Solving Quadratic Equations.

We have seen how the properties of maximum value, minimum value, and symmetry of a quadratic function are related to the coefficients a, b, and c in the equation of the function. This relation can be a powerful tool for the study of quadratic functions associated with many real world situations. We illustrate this power by returning our attention to the quadratic functions associated with freely falling bodies.

Example 7-9-1

Find the maximum displacement of the ball whose displacement is described by the function:

$$d = \{(t, d(t)) | d = -16t^2\}$$

Solution:

(1) $a = -16 < 0$ so the function has maximum value.

(2) $b = 0$ and $c = 0$.

(3) The ordinate of the critical point is:

$$\frac{4ac - b^2}{4a} = \frac{4(-16)(0) - (0)^2}{4(-16)} = 0$$

Hence the maximum displacement is 0. This agrees with the fact that the ball was dropped straight down from the roof top.

Example 7-9-2

Find the maximum displacement of the ball whose displacement is described by the function:

$$d = \{(t, d(t) | d = -16t^2 + 80t\}$$

Solution:

(1) $a = -16 < 0$ so the function has a maximum value.

(2) $b = 80$ and $c = 0$.

(3) The ordinate of the critical point is:

$$\frac{4ac - b^2}{4a} = \frac{4(-16)(0) - (80)^2}{4(-16)} = +100$$

Hence the maximum displacement of the ball is +100. This means the highest point reached by the ball is +100 feet above the roof top.

Exercise 7-9-3

Find the maximum displacement of the ball whose motion is described by the function:

$$d = \{(t,d(t)) | d = -16t^2 + 80t - 96\}$$

The previous examples illustrate how the coefficients a, b and c can be used to determine the maximum or minimum value of a quadratic function. A second and more interesting problem associated with these functions might be to determine the time elapsed when the ball reaches its maximum displacement. In addition we might want to determine the time elapsed before the ball attains any predetermined displacement. The following examples are illustrative of this type of problem.

Example 7-9-4

If the ball moves with a displacement described by the function

$$d = \{(t,d(t)) | d = -16t^2 + 80t - 96 \},$$

find the time elapsed when the ball reaches its maximum displacement.

Solution:

(1) First find the maximum displacement

$$a = -16,\ b = 80,\ c = -96$$

$$\begin{aligned} \frac{4ac - b^2}{4a} &= \frac{4(-16)(-96) - (80)^2}{4(-16)} \\ &= -96 - \frac{6400}{-64} \\ &= -96 + 100 \\ &= +4 \text{ (maximum displacement)} \end{aligned}$$

(2) Determine the time elapsed by substituting the maximum displacement $d = 4$ into the set selector equation.

$$\begin{aligned} d = 4 \text{ and } d &= -16t^2 + 80t - 96 \\ 4 &= -16t^2 + 80t - 96 \\ 0 &= -16t^2 + 80t - 100 \quad \text{Substitution, APE} \end{aligned}$$

Since the left hand side of this equation is zero, the multiplicative property of equality can be used to simplify the equation. Multiply both sides by (-1/16).

$$0 = t^2 - 5t + \frac{100}{16} \qquad \text{MPE}$$

$$0 = t^2 - 5t + \frac{25}{4} \qquad \text{RTPE}$$

$$0 = t^2 - 5t + (5/2)^2 \qquad \text{RTPE}$$

The expression $t^2 - 5t + (5/2)^2$ is a perfect square quadratic. Hence the following replacement is made:

$$0 = (t - 5/2)^2 \qquad \text{RTPE}$$

We can see by inspection that the solution set for this equation is

$$\{5/2\}$$

Hence the time elapsed when the ball reaches its maximum displacement of +4 feet is 2 1/2 second.

Example 7-9-5

If the equation of motion is $d = -16t^2 + 80t - 96$, find the time elapsed when $d = -192$.

Solution:

$$d = -16t^2 + 80t - 96$$

$$-192 = -16t^2 + 80t - 96 \qquad \text{Substitution}$$

$$0 = -16t^2 + 80t + 96 \qquad \text{APE}$$

$$0 = t^2 - 5t - 6 \qquad \text{MPE}$$

The expression $t^2 - 5t - 6$ is not a perfect square quadratic. Hence we must use the replacement transformation principle for equations and complete the square on this expression.

$$0 = t^2 - 5t - 6$$

$$0 = t^2 - 5t + (5/2)^2 - 6 - (5/2)^2 \qquad \text{RTPE}$$

To complete the square we add 0, written as $(\frac{5}{2})^2 - (\frac{5}{2})^2$, to the right side.

$$0 = (t - \tfrac{5}{2})^2 - (6 + (\tfrac{5}{2})^2) \qquad \text{RTPE}$$

$$0 = (t - \tfrac{5}{2})^2 - (6 + \tfrac{25}{4}) \qquad \text{RTPE}$$

$$0 = (t - \tfrac{5}{2})^2 - (\tfrac{24}{4} + \tfrac{25}{4}) \qquad \text{RTPE}$$

$$0 = (t - \tfrac{5}{2})^2 - \tfrac{49}{4} \qquad \text{RTPE}$$

$$0 = (t - \tfrac{5}{2})^2 - (\tfrac{7}{2})^2 \qquad \text{RTPE}$$

We are still unable to determine the solution set by inspection. However, the expression $(t - \frac{5}{2})^2 - (\frac{7}{2})^2$ is of the form $x^2 - y^2$. In Chapter 3 we had the theorem,

$\forall x \forall y \quad x^2 - y^2 = (x + y)(x - y)$, so we continue the transformation.

$$0 = (t - \tfrac{5}{2})^2 - (\tfrac{7}{2})^2$$

$$0 = [(t - \tfrac{5}{2}) - \tfrac{7}{2}]\ [(t - \tfrac{5}{2}) + \tfrac{7}{2}] \qquad \text{RTPE}$$

$$0 = (t - 6)(t + 1) \qquad \text{RTPE}$$

$$t - 6 = 0 \text{ or } t + 1 = 0 \qquad \text{Theorem 2-5-18}$$

$$t = 6 \text{ or } t = -1 \qquad \text{APE}$$

We can check our results by substituting the elements in the solution set $\{6, -1\}$ into the original equation.

$-192 = -16t^2 + 80t - 96$	$-192 = -16t^2 + 80t - 96$
$-192 = -16(6)^2 + 80(6) - 96$	$-192 = -16(-1)^2 + 80(-1) - 96$
$-192 = -576 + 480 - 96$	$-192 = -16 - 80 - 96$
$-192 = -192$	$-192 = -192$
OK	OK

Both solutions satisfy the equation . If you review the original situation corresponding to the motion $d = -16t^2 + 80t - 96$, you will remember that the ball was thrown upward with an initial velocity of +80 ft/sec from an initial displacement of -96 feet. Hence, the ball can reach a displacement of -192 only after time has elapsed and the ball is on the return trip toward the ground. Hence, the displacement is $d = -192$ only when $t = 6$. Can you attach any significance to the solution $t = -1$?

Exercise 7-9-6

1. If the motion is described by the equation $d = -16t^2 + 80t - 96$, how much time has elapsed when $d = -32$ feet?

2. If the motion is described by the equation $d = -16t^2 + 80t$, how much time has elapsed when $d = +96$ feet?

We have previously defined "solving an equation" to mean "finding the replacements for the variables which make the equation a true statement." Equations of the form $y = ax^2 + bx + c$ have infinite sets of ordered pairs (x,y) as their solution sets. However, when a specified value of the range of such a function is substituted for y in the equation, the problem becomes one of finding the corresponding members of the domain which map into that element of the range. This process is called "solving quadratic equations".

Example 7-9-7

In the function $\{(x,y) | y = 2x^2 - 2x - 1\}$, find the elements in the domain which map into the element 3 in the range.

Solution

$y = 2x^2 - 2x - 1$ and $y = 3$

$3 = 2x^2 - 2x - 1$ Substitution

$0 = 2x^2 - 2x - 4$ APE

$0 = x^2 - x - 2$ MPE

Now complete the square

$0 = x^2 - x + (1/2)^2 - (1/2)^2 - 2$ RTPE

$0 = (x - 1/2)^2 - (9/4)$ RTPE

$0 = (x - 1/2)^2 - (3/2)^2$ RTPE

$0 = [(x - 1/2) - 3/2][(x - 1/2) + 3/2]$ RTPE

$0 = (x - 2)(x + 1)$ RTPE

$x - 2 = 0$ or $x + 1 = 0$ Theorem 2-5-18

$x = 2$ or $x = -1$

The elements $\{2, -1\}$ from the domain map into the element $\{3\}$ in the range.

Example 7-9-8

Find the solution of $3x^2 - 2x - 1 = 0$

Note: We must find those values of x for which $y = 0$ in the function

$$\{(x,y) | y = 3x^2 - 2x - 1\}$$

Solution

$0 = 3x^2 - 2x - 1$	
$0 = x^2 - 2/3x - 1/3$	MPE
$0 = x^2 - 2/3x + (1/3)^2 - 1/3 - (1/3)^2$	RTPE
$0 = (x - 1/3)^2 - (2/3)^2$	RTPE
$0 = [(x - 1/3) - 2/3][(x - 1/3) + 2/3]$	RTPE
$0 = (x - 1)(x + 1/3)$	RTPE
$x - 1 = 0$ or $x + 1/3 = 0$	Theorem 2-5-18
$x = 1$ or $x = -1/3$	APE

The solution set is $\{1, -1/3\}$

Exercise 7-9-9

Solve the following quadratic equations for x.

1. $x^2 - 5x + 6 = 0$
2. $2x^2 + 3x = 0$
3. $2x^2 - 3x - 5 = 0$
4. $25x^2 = -10x - 1$
5. $2x^2 + 7x + 2 = 6$
6. $5x^2 - 5x = 2$

All of the previous examples and exercises in this section deal with quadratic equations which have rational solutions. However, many quadratic equations, which have irrational solutions also exist. The procedure for solving such equations by completing the square works equally well.

Example 7-9-10

Solve the equation $4x^2 - x - 2 = 0$ for x.

Solution

$$4x^2 - x - 2 = 0$$

$$x^2 - 1/4x - 1/2 = 0 \qquad \text{MPE}$$

$$x^2 - 1/4x + 1/64 - 1/64 - 1/2 = 0 \qquad \text{RTPE}$$

$$(x - 1/8)^2 - \frac{33}{64} = 0 \qquad \text{RTPE}$$

$$(x - 1/8)^2 - \left(\frac{\sqrt{33}}{8}\right)^2 = 0 \qquad \text{RTPE}$$

$$\left[(x - 1/8) + \frac{\sqrt{33}}{8}\right]\left[(x - 1/8) - \frac{\sqrt{33}}{8}\right] = 0 \qquad \text{RTPE}$$

$$x - 1/8 + \frac{\sqrt{33}}{8} = 0 \text{ or } x - 1/8 - \frac{\sqrt{33}}{8} = 0 \qquad \text{Theorem 2-5-18}$$

$$x = \frac{1 - \sqrt{33}}{8} \quad \text{or } x = \frac{1 + \sqrt{33}}{8}$$

Both solutions to the equation are irrational numbers. We prefer to leave the solutions in radical form rather than to give approximate decimal values for the solutions. The solutions can be verified by showing that their substitution into the original equation yields a true statement

$$4x^2 - x - 2 = 0$$

$$4\left(\frac{1 - \sqrt{33}}{8}\right)^2 - \left(\frac{1 - \sqrt{33}}{8}\right) - 2 = 0$$

$$4\left(\frac{1 - 2\sqrt{33} + 33}{64}\right) - \left(\frac{8 - 8\sqrt{33}}{64}\right) - 2 = 0$$

$$\frac{4 - 8\sqrt{33} + 132}{64} - \left(\frac{8 - 8\sqrt{33}}{64}\right) - \frac{128}{64} = 0$$

$$0 = 0$$

Exercise 7-9-11

Verify that $\frac{1 + \sqrt{33}}{8}$ is an element of the solution set of the equation $4x^2 - x - 2 = 0$.

Exercise 7-9-12

Find the solutions to the following equations. Verify the solutions by substitution.

1. $5x^2 - 2x - 1 = 0$
2. $2x^2 - 5x + 5 = 7$
3. $4x^2 + 9x = -4$
4. $2x^2 - 3x - 8 = 0$
5. $2x^2 + 3x + c = 0$
6. $2x^2 + bx + c = 0$
7. $Ax^2 + bx + c = 0$

The "completing the square" method of solving quadratic equations is a laborious process. We can simplify the process of solving quadratic equations by developing a formula which can be used to solve quadratic equations.

We begin by completing the square on the general quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0 .$$

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

$$x^2 + (b/a)x + (c/a) = 0 \qquad \text{MPE}$$

$$x^2 + (b/a)x + (b/2a)^2 + (c/a) - (b/2a)^2 = 0 \qquad \text{RTPE}$$

$$[x + (b/2a)]^2 + \frac{4ac - b^2}{4a^2} = 0 \qquad \text{RTPE}$$

$$[x + (b/2a)]^2 - [\frac{b^2 - 4ac}{4a^2}] = 0 \qquad \text{RTPE}$$

$$[x + (b/2a) - \frac{\sqrt{b^2 - 4ac}}{2a}] \cdot [x + (b/2a) + \frac{\sqrt{b^2 - 4ac}}{2a}] = 0, b^2 - 4ac \geq 0 \qquad \text{RTPE}$$

$$[x + \frac{b - \sqrt{b^2 - 4ac}}{2a}] \cdot [x + \frac{b + \sqrt{b^2 - 4ac}}{2a}] = 0 \qquad \text{RTPE}$$

$$x + \frac{b - \sqrt{b^2 - 4ac}}{2a} = 0 \text{ or } x + \frac{b + \sqrt{b^2 - 4ac}}{2a} = 0 \qquad \text{Theorem 2.75}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{RTPE}$$

We have just proved the following theorem.

Theorem 7-9-13

$$\forall a \neq 0, \forall b, \forall c, \forall x, \text{ such that } b^2 - 4ac \geq 0,$$

$$ax^2 + bx + c = 0$$

If and only if

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{or } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

A part of this theorem is generally referred to as the "Quadratic Formula" and is written:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It can be used to solve any equation of the form $ax^2 + bx + c = 0$ when $b^2 - 4ac \geq 0$.

Example 7-9-14

Solve the equation $2x^2 + 7x + 12 = 8$ by use of the quadratic formula.

Solution:

$$2x^2 + 7x + 12 = 8$$

First we must transform the equation into the form $ax^2 + bx + c = 0$. Hence

$$2x^2 + 7x + 4 = 0$$

In this equation $a = 2$, $b = 7$ and $c = 4$. We also find that $b^2 - 4ac > 0$ as follows:

$$b^2 - 4ac = 7^2 - 4(2)(4)$$

$$b^2 - 4ac = 49 - 32$$

$$b^2 - 4ac = 17 \geq 0$$

Hence $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-7 \pm \sqrt{17}}{4}$

The solution set is $\{\frac{-7 + \sqrt{17}}{4}, \frac{-7 - \sqrt{17}}{4}\}$

Exercise 7-9-15

Solve the following equations by use of the quadratic formula.

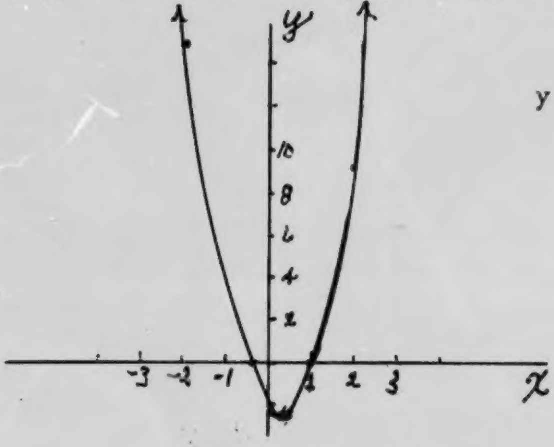
1. a. $5x^2 + 9x + 4 = 0$
   b. $x^2 + 2x - 4 = 0$
   c. $3x^2 = 2x + 2$
   d. $4x^2 - 3x - 8 = -6$
   e. $x^2 + x - 1 = 0$
   f. $10x^2 - x - 3 = 0$
   g. $x^2 + 4x - 6 = 0$
   h. $4x^2 - 5x + 2 = 0$
   i. $3x^2 + 7x - 2 = 0$
   j. $x^2 + 1 = 0$

2. Check your answers to problems b, h and j by substitution.

3. Write a computer program that will cause the computer to solve quadratic equations by use of the quadratic formula. Use the computer to check your answers to problems a - j above.

In those quadratic equations encountered so far which had real number solutions, the solutions were elements from the domain of the corresponding function which mapped into the real number zero.

In the function $\{(x,y)|y = 3x^2 - 2x - 1\}$ the elements $\{1, -1/3\}$ from the domain are associated with the element $\{0\}$ from the range. In other words $(1,0)$ and $(-1/3,0)$ are elements of the function. A graph of the function is shown in Figure 7-9-16



$y = 3x^2 - 2x - 1$

Figure 7-9-16

Examination of Figure 7-9-16 reveals that the solutions $\{1,-1/3\}$ for the equation $3x^2 - 2x - 1 = 0$ are the x coordinates of the points where the function $\{(x,y|y = 3x^2 - 2x - 1\}$ intersects the x-axis.

A function of the form
$\{(x,y)|y = ax^2 + bx + c\}$
crosses the x-axis at the
point $(m,0)$ and $(n,0)$
if and only if

$\{m,n\}$ is the solution set
of the equation $ax^2 + bx + c = 0$.

We remember from our work in section 7.8 that the graphs of many quadratic functions do not cross the x-axis. This means that none of the elements in the domain of such a function will map into zero in the range. Such a function is shown in Figure 7-9-17.

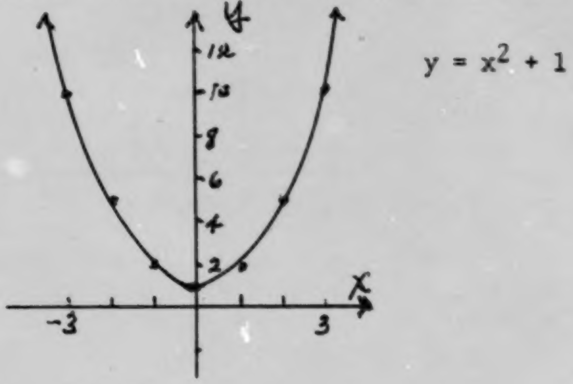


Figure 7-9-17

In this function we see that no real numbers from the domain map into the real number zero. Hence, the equation $x^2 + 1 = 0$ has no real number solutions. In order to be able to find solutions for equations of this kind, we must extend our number system beyond the set of reals. We will do this in the next section of this chapter.

Review Problem Set 7-9-18

Factor, as completely as possible, the following over the field of real numbers.

1. $ax + bx$
2. $m^2x + n^2x$
3. $2my - 8m^2y$
4. $4a^2 - 1$
5. $1 - 16b^4$
6. $9x^4 - 16$
7. $25x^2 + 30x + 9$
8. $16x^2 - 24x + 9$
9. $2x^2 - 18$
10. $9s^2 - 9s^4$
11. $ax + bx + ay + by$
12. $2ax + 3 + x + 6a$
13. $2x^3 - 3x^2 + 2x - 3$
14. $8x^3 + 1$
15. $r^3 + s^3$
16. $125y^3 - 1$
17. $1 - 27x^3$
18. $x^2 - 4x + 3$
19. $2a^2 - 5a + 2$
20. $15t^2 + 5t - 10$
21. $3x^3 + 18x^2 - 48x$
22. $16x^3y - 24x^2y^2 - 64xy^3$
23. $(x^2 - 1)^2 - 8(x^2 - 1) + 12$
24. $2\cos^2 y + 3\cos y + 1$
25. $\sin^2 x - \sin^2 x \cos^2 x + 2 \sin x \cos x + \cos^2 x$

7-10 A New Set of Numbers with New Operations

A new set of numbers C and the operations on this set will be defined as follows:

Definitions 7-10-1

A. The definition of the set.

$C = \{(x,y) | x \epsilon R,\ y \epsilon R\}$

B. Operations on Set C

For any two ordered pairs, (a,b) and $(c,d) \epsilon C$:

(1) Addition: $(a,b) + (c,d) = (a + c,\ b + d)$ where the + sign on the right is real number addition.

(2) Multiplication: $(a,b) \cdot (c,d) = (ac - bd,\ ad + bc)$ where the operations on the right are real number operations.

C. Equality in C

$(a,b) = (c,d)$ if and only if $a = c$ and $b = d$, where the equal signs on the right indicate real number equality.

To write a number in terms of pairs of numbers is really not as strange as it might seem. We might have chosen to write rational numbers in the form (a,b), $b \neq 0$ instead of the conventional form $\frac{a}{b}$, $b \neq 0$. Then the addition of two rational numbers, $\frac{a}{b}$, $b \neq 0$ and $\frac{c}{b}$, $b \neq 0$ could have been defined as:

$$(a,b) + (c,b) = (a + c,b).$$

Exercise Set 7-10-2

1. Write the general definition of addition for rational numbers when they are written in the form (a,b).

2. Define the operation of multiplication for two rational numbers, when written in the form (a,b).

3. Write a computer program that would cause the computer to print out the sum of two numbers of C in the following format:

$$(a,b) + (c,d) = (x,y)$$

4. Write a computer program that would cause the computer to print out the product of two numbers in C in the format:

$$(a,b)(c,d) = (x,y)$$

5. Using the above program have the computer print out the sum of these pairs of numbers in the following format:

$$(a,b) + (c,d) = (x,y)$$

   a. (4,3)(-1,0)

   b. (-2,5)(3,1)

   c. (3.061, -2.673)(-3.061, 2.673)

   d. (0,1)(1,0)

   e. (-1,2)(-1,2)

   f. $(1,1)(\frac{1}{2}, -\frac{1}{2})$

6. Using the above program for multiplication on C, find the product of each of the pairs of numbers in #5. Print the product in the format:

$$(a,b)(c,d) = (x,y)$$

7. Are any of the pairs of numbers in Problem 5 equal?

Let us now investigate some subsets of C. The following exercises deal with the subset $S_1 = \{(x,y) | y = 0\}$.

Exercise Set 7-10-3

1. Complete the following statements on S_1.

   Write out or have the computer print out the statements in the following format:

$$(a,b) + (c,d) = (x,y)$$

   a) (5,0) + (3,0) =

   b) (-7,0) + (7,0) =

   c) (3,0) + ((8,0) + (-2,0)) =

   d) (4.8721,0) + (0,0) =

   e) (16.123,0) + (8.17,0) =

   f) (873.721,0) + (-873.721,0) =

   g) (17,0) + (1,0) =

   h) (0,0) + (-723.4,0) =

   i) (1,0) + (17,0) =

   j) ((3,0) + (8,0)) + (-2,0) =

2. On the basis of the results in the above exercise answer these questions.

   a) Does closure in addition seem to hold on S_1?

   b) Is there an additive identity element?

   If so, name it. If not, present a counter example.

   c) Does each element seem to have an additive inverse in S_1? If so, what is the additive inverse of (a,0). If not, offer a counter example.

   d) Do the associative and commutative properties seem to hold?

3. Perform the indicated multiplications using the definition of multiplication on C or the computer program you wrote in the previous exercise set. Write out or print out the results in (a,b)(c,d) = (x,y) form.

   a) (2,0)(3,0) =

   b) (-6,0)(5,0) =

   c) (8.732,0)(-8.732,0) =

   d) (5,0)((-3,0)(4,0)) =

   e) (3,0)(2,0) =

   f) (-17,0)(1,0) =

   g) $(7,0)(\frac{1}{7}, 0) =$

   h) ((5,0)(-3,0))(4,0) =

   i) (1,0)(-6.713,0) =

   j) (-1,0)(-10,0) =

   k) $(1,0)^2 =$

   l) $(-1,0)^2 =$

   m) $(a,0)^2 =$

   n) $(-a,0)^2 =$

4. Based on the results obtained in Problem 3 above, answer the questions of #2 with respect to multiplication.

5. a) Show whether (3,0) +((2,0)(1,0)) does or does not equal ((3,0) + (2,0))((3,0) + (1,0)).

   b) Show whether (3,0)((2,0) + (1,0)) does or does not equal ((3,0)(2,0)) + ((3,0)(1,0)).

6. Define subtraction in S_1.

7. Can you define division in S_1 in 2 ways? If so, do so.

8. Does S_1 remind you of a number system familiar to you? If so, which one and why? If it doesn't, look again.

The purpose of the preceding exercise was to show that the real numbers are imbedded in the set C. For each real number $x \varepsilon R$, there exists the ordered pair $(x,0) \varepsilon C$ so that the operations of addition and multiplication in R correspond to addition and multiplication as defined in C. That is

$$\begin{tikzcd}
(a,0) \arrow[d] & + & (b,0) \arrow[d] & = & (a+b,0) \arrow[d] \\
a & + & b & = & a+b
\end{tikzcd}
\qquad
\begin{tikzcd}
(a,0) \arrow[d] & \cdot & (c,0) \arrow[d] & = & (ac,0) \arrow[d] \\
a & \cdot & c & = & ac
\end{tikzcd}$$

Example:

$$\begin{tikzcd}
(2,0) \arrow[d] & + & (-\frac{1}{2}, 0) \arrow[d] & = & (\frac{3}{2}, 0) \arrow[d] \\
2 & + & -\frac{1}{2} & = & \frac{3}{2}
\end{tikzcd}
\qquad
\begin{tikzcd}
(2,0) \arrow[d] & \cdot & (-\frac{1}{2}, 0) \arrow[d] & = & (-1,0) \arrow[d] \\
2 & \cdot & -\frac{1}{2} & = & -1
\end{tikzcd}$$

Two sets of numbers that are related in this manner are called isomorphic sets.

Now we will consider another subset of C, $S_2 = \{(x,y) | x = 0\}$. Some elements of this set are $(0,3)$, $(0, -\frac{1}{2})$, etc.

Exercise Set 7-10-4

1. In a manner similar to Exercise 7-10-3 determine if S_2 is a field. Use elements of your own choosing. If it is not a field, indicate which properties fail.

S_2 is sometimes refered to as the set of imaginary numbers.

Now we will consider some exercises in C, $C = \{(x,y) | x,y \varepsilon R\}$.

Exercise 7-10-5

Complete the following statements by writing them in the form.

$$(a,b)(c,d) = (x,y)$$

1. $(4,5)(2,3) = ?$

2. $(5,3)(1,0) = ?$

3. $(6.876, -53.16)(1,0) = ?$

4. $(0,5)(0, -\frac{5}{25}) = ?$

5. $(3,0)(\frac{3}{9},0) = ?$

6. $(4,2)(\frac{4}{20}, -\frac{2}{20}) = ?$

7. $(0,2)(1,0) = ?$

8. $(5,0)(1,0) = ?$

9. On the basis of your work in #1-#8 would you venture a guess as to the multiplicative identity in C, multiplicative inverse of $(a,b) \varepsilon C$?

10. Choose some ordered pairs from C and by example show that each of the other field properties seems to hold.

At this point it would seem reasonable to conclude that C is a field. In fact $(C, +, \cdot)$ is a field normally referred to as the field of complex numbers. We will not attempt to prove that it is.

The complex number field differs from the real field in that the "less than" relationship does not hold in C. Therefore, the complex field is referred to as a non-ordered field while the real field is an ordered field.

The elements of the real field are ordered and can be graphed on a number line. For any pair of real numbers one number will be to the left of the other and therefore, "less than" the other number.

Complex numbers can not be positioned on the number line but in more advanced study of complex numbers they are graphed as ordered pairs in the Cartesian plane and no definition of "less than" is made.

An interesting activity will be to solve equations in C. In order to do this it is necessary to know the general form of the multiplicative inverse of $(a,b) \varepsilon C$. If you did not discover it in #10 above, it is

$$\left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2}\right) \varepsilon C$$

Example 7-10-6: Solve $(1,-2)(x,y) = (2,1)$ for (x,y)

1. $(1,-2)(x,y) = (3,1)$	1. Given equation.
2. $(\frac{1}{5}, \frac{2}{5})(1,-2)(x,y) = (\frac{1}{5}, \frac{2}{5})(3,1)$	2. Multiplying both sides of the equation by the multiplicative inverse in C of (1,-2)
3. $(\frac{1}{5} + \frac{4}{5}, \frac{2}{5} - \frac{2}{5})(x,y) = (\frac{3}{5} - \frac{2}{5}, \frac{1}{5} + \frac{6}{5})$	3. Definition of multiplication in C
4. $(1,0)(x,y) = (\frac{1}{5}, \frac{7}{5})$	4. Arithmetic
5. $(x - 0, y + 0) = (\frac{1}{5}, \frac{7}{5})$	5. Definition of multiplication in C
6. $(x,y) = (\frac{1}{5}, \frac{7}{5})$	6. Additive identity of the reals

Check: $(1,-2)\ (\frac{1}{5}, \frac{7}{5}) = (\frac{1}{5} + \frac{14}{5}, \frac{-2}{5} + \frac{7}{5}) = (3,1)$

Problem Set 7-10-7

Solve for (x,y)

1. $(x,y) + (1,1) = (0,0)$
2. $(x,y) + (2,-1) = (0,3)$
3. $(0,2)(x,y) = (2,2)$
4. $(1,2)(x,y) + (1,0) = (-2,0)$
5. $(1,-1)(x,y) + (x,y) = (3,1)$
6. $(5,-3) + (x,y) = (1,2)$
7. $(-6,2) + (x,y) = (3,-1)$
8. $(2,3)(x,y) = (1,0)$
9. $(4,2)(x,y) = (1,0)$

10. The general form of the multiplicative inverse of any ordered pair $(a,b) \epsilon C$ can be found by solving the equation $(a,b)(x,y) = (1,0)$ for (x,y). You should arrive at:

$$x = \frac{a}{a^2 + b^2} \quad \text{and} \quad y = \frac{-b}{a^2 + b^2}$$

11. Solve each of the following equations for (a,b).

    a) $(a,b)^2 = (1,0)$

    b) $(a,b)^2 = (-1,0)$

    c) $(a,b)^2 = (0,1)$

    d) $(a,b)^2 = (0,-1)$

12. Complete:

    a) $(1,0)^2 =$

    b) $(-1,0)^2 =$

    c) $(0,1)^2 =$

    d) $(0,-1)^2 =$

13. Write a computer program which will print the powers of $(0,1)^n$ and $(0,-1)^n$ for $n = 1$ to 10.

Example 7-10-8

We are now ready to return to the problem of solving $x^2 + 1 = 0$. Lets assume there is a solution $(a,b) \epsilon C$. Since $1 \epsilon R \leftrightarrow (1,0) \epsilon C$ and $0 \epsilon R \leftrightarrow (0,0)\ C$ the equation $x^2 + 1 = 0 \leftrightarrow (a,b)^2 + (1,0) = (0,0)$

Solution:

$$(a,b)^2 + (1,0) = (0,0)$$

$$(a^2 - b^2, + 2ab) + (1,0) = (0,0)$$

$$(a^2 - b^2 + 1, 2ab) = (0,0)$$

$$a^2 - b^2 + 1 = 0 \text{ and } 2ab = 0$$

$$2ab = 0 \rightarrow a = 0$$

$$0^2 - b^2 + 1 \rightarrow b = 1$$

$$b = +1 \text{ or } - 1$$

So we must check $(0,1)$ and $(0,-1)$.

Let $(x,y) = (0,1)$: $(0,1)^2 + (1,0) = (0,0)$

$$(-1,0) + (1,0) = (0,0)$$

$$(0,0) = (0,0)$$

Let $(x,y) = (0,-1)$: $(0,-1)^2 + (1,0) = (0,0)$

$$(-1,0) + (1,0) = (0,0)$$

$$(0,0) = (0,0)$$

We see that $x^2 + 1 = 0$ has solutions $(0,1)$ and $(0,-1)$

Example 7-10-9

Let's try solving $2x^2 + x + 1 = 0$ which has no real number solution by expressing it as $(2,0) \cdot (a,b)^2 + (a,b) + (1,0) = (0,0)$ where $(a,b) \varepsilon C$.

Solutions:

$$(2,0)(a,b)^2 + (a,b) + (1,0) = (0,0)$$

$$(2,0)(a^2 - b^2, 2ab) + (a,b) + (1,0) = (0,0)$$

$$(2a^2 - 2b^2, 4ab) + (a,b) + (1,0) = (0,0)$$

$$(2a^2 - 2b^2 + a + 1, 4ab + b) = (0,0)$$

$$2a^2 - 2b^2 + a + 1 = 0 \text{ and } 4ab + b = 0$$

$$2a^2 - 2b^2 + a + 1 = 0 \text{ and } b(4a + 1) = 0$$

$$2a^2 - 2b^2 + a + 1 = 0 \text{ and } [b = 0 \text{ or } 4a + 1 = 0]$$

If $b = 0$ then (a,b) is real and there are no solutions, If $4a + 1 = 0$ then $a = -\frac{1}{4}$

$$2(-\frac{1}{4})^2 - 2b^2 + -\frac{1}{4} + 1 = 0$$

$$\frac{1}{8} - 2b^2 + \frac{3}{4} = 0$$

$$-2b^2 = -\frac{7}{8}$$

$$b = \pm\sqrt{\frac{7}{16}} = \pm\frac{\sqrt{7}}{4}$$

Let's check to see if $(-\frac{1}{4}, \frac{\sqrt{7}}{4})$ and $(-\frac{1}{4}, -\frac{\sqrt{7}}{4})$ are solutions of the original equation.

Let $(a,b) = (-\frac{1}{4}, \frac{\sqrt{7}}{4})$: $(2,0)(-\frac{1}{4}, \frac{\sqrt{7}}{4})^2 + (-\frac{1}{4}, \frac{\sqrt{7}}{4}) + (1,0) = (0,0)$

$$(2,0)(\frac{1}{16} - \frac{7}{16} - \frac{\sqrt{7}}{8}) + (-\frac{1}{4}, \frac{\sqrt{7}}{4}) + (1,0) = (0,0)$$

$$(-1,0) + (1,0) = (0,0)$$

$$(0,0) = (0,0)$$

The rest of the check is #1 below.

Exercise 7-10-10

1. Show that $(-\frac{1}{4}, -\frac{\sqrt{7}}{4})$ is also a root of $2x^2 + x + 1 = 0$.

2. Find the solutions of each of the following equations:

   (a) $x^2 + 3 = 0$

   (b) $3x^2 - 3x + 1 = 0$

   (c) $2x^2 + 3x + 2 = 0$

7-11 The Rectangular $(x + yi)$ Form of the Complex Numbers.

In the previous set of exercises we were able to solve equations in C which did not have solutions in R. The process of solving was cumbersome; therefore, we will introduce a new notation for the elements in C that will simplify the process.

Remember that we have established the fact that the reals are a subset of C, e.g., $(x,0) = x$, $x \in R$. If we let the letter i represent the ordered pair $(0,1)$ we will show that the ordered pair (x,y) can be written as $x + yi$.

$(x,y) = (x,0) + (0,y)$	Definition of Addition in C
$(x,0) = x$, $x \in R$	Isomorphism
$(x,y) = x + (0,y)$	Substitution
$(0,y) = (y,0)(0,1)$	Definition of Multiplication in C
$(x,y) = x + (y,0)(0,1)$	Substitution
$(y,0) = y$, $y \in R$	Isomorphism
$(x,y) = x + y(0,1)$	Substitution
$(x,y) = x + yi$	Substitution

Example 7-11-0

$(7,3) = 7 + 3i$

$(9,-5) = 9 - 5i$

The symbol $i = (0,1)$ has the property that $i^2 = -1$. In previous exercises we showed that $(0,1)^2 = (-1,0)$ and $(-1,0) = -1 \in R$.

The advantage of the $a + bi$ form of complex numbers is that we can find sums and products without remembering the definitions of addition and multiplication in C. All we need to remember is that $i^2 = -1$ and the operations in R.

Definition 7-11-1: Complex Numbers

$(x + yi)$, $x \in R$, $y \in R$ and $i = (0,1)$ is defined to be the rectangular form of elements of C.

Example 7-11-2

If we treat the $x + yi$ form of the complex numbers as binomial expressions in R we can perform the operations as follows:

$$(2 + 3i) + (8 - 5i) = (2 + 8) + (3 - 5)i = 10 - 2i$$

and

$$\begin{aligned}(2 + 3i)(8 - 5i) &= (2 + 3i)8 + (2 + 3i)(-5i) \\ &= 16 + 24i - 10i - 15i^2 \\ &= 16 + 24i - 10i - 15(-1) \\ &= 31 + 14i\end{aligned}$$

Exercise 7-11-3

Verify the results of the above example by using the ordered pair form definitions.

Formal definitions in $x + yi$ form are as follows:

Definition 7-11-4: Addition in C for $x + yi$ notation.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Definition 7-11-5: Multiplication in C for $x + yi$ notation

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Definition 7-11-6: Equality of Complex Numbers in $x + yi$ form.

$(a + bi) = (c + di)$ if and only if $a = c$ and $b = d$.

Definition 7-11-7: Conjugates in C.

Two complex numbers of the form $a + bi$ and $a - bi$ are said to be conjugates of each other.

Exercise 7-11-8

1. Show that $((a + bi) + (a - bi)) \varepsilon R$

2. Show that the product of conjugate pairs of complex numbers $((a + bi) \cdot (a - bi)) \varepsilon R$.

3. Using the properties discussed in this section carry out the indicated operations.

   a. $(5 - 6i) + (4 - i)$

   b. $(-2 + 3i) - (6 + 2i)$

   c. $(3 + 7i)(11 - 5i)$

   d. $(4 - 7i)(4 + 7i)$

   e. $(2i)^3$

   f. $(7i)(2i + 21)$

   g. $-\frac{13}{2}(11i - 5i)$

   h. $2 + (3i)^3$

   i. $(2 + 3i)^3$

   j. $i^2 + i^4 + i^6$

   k. $6(2 + i) + 3(3 + 2i)$

   l. $(\frac{\sqrt{3}}{2} + \frac{1}{2}i)^2$

4. Write each of the following as a complex number in the form $a + bi$.

   a. $9i + \frac{1}{i^2}$

   b. $-\frac{1}{2i} + 6$

   c. $\frac{4}{i^4}$

   d. $3 - \frac{3}{2i^{10}}$

Problem Set 7-11-9

Using the rectangular form of the complex numbers, show that C is a field. Two of the field properties are verfied for you.

1. Closure of Multiplication in C

   $\forall (a + bi) \in C, \forall (c + di) \in C.$

$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$	Definition of Multiplication in C.
$(ac - bd) \varepsilon R$	$a,c,b,c,d \in R$ Closure for subtraction and multiplication in R.
$(ad + bc) \varepsilon R$	$a,d,b,c, \in R$ Closure for addition and multiplication in R

$\therefore\ (ac - bd) + (ad + bc)i \in C$ Definition of elements in C of the form $x + yi$

2. The Commutative Property for Addition in C

   $\forall\ (a + bi) \in C,\ \forall\ (c + di) \in C$

   $(a + bi) + (c + di) = (a + c) + (b + d)i$ Definition of Addition in C

   $= (c + a) + (d + b)i$ CPA for R

   $= (c + di) + (a + bi)$ Definition of Addition in C

3. Closure Addition in C

   $\forall\ (a + bi) \in C,\ \forall\ (c + di) \in C$

   $(a + bi) + (c + di) = ((a + c) + (b + d)i) \in C$

4. Associative Property of Addition in C

   $\forall\ (a + bi) \in C,\ \forall\ (c + di) \in C,\ \forall\ (e + fi) \in C$

   $((a + bi) + (c + di)) + (e + fi) = (a + bi) + ((c + di) + (e + fi))$

5. Addition Identity Element in C

   $\forall\ (a + bi) \in C,\ \exists\ (0 + 0i) \in C$ such that

   $(a + bi) + (0 + 0i) = a + bi$

6. Associative Property of Multiplication in C

   $\forall\ (a + bi) \in C,\ \forall\ (c + di) \in C,\ \forall\ (e + fi) \in C$

   $((a + bi) \cdot (c + di)) \cdot (e + fi) = (a + bi)((c + di) \cdot (e + fi))$

7. Commutative Property of Multiplication in C

   $\forall\ (a + bi) \in C,\ \forall\ (c + di) \in C$

   $(a + bi)(c + di) = (c + di)(a + bi)$

8. Multiplicative Identity Element in C

   $\forall\ (a + bi) \in C\ \exists\ (1 + 0i) \in C$, such that

   $(1 + 0i)(a + bi) = (a + bi)(1 + 0i) = a + bi$

9. Distribution of Multiplication over Addition in C

$\forall\ (a + bi)\epsilon C,\ \forall\ (c + di)\epsilon C,\ \forall\ (e + fi)\epsilon C$

$(a + bi)((c + di) + (e + fi)) = (a + bi)(c + di) + (a + bi)(e + fi)$

10. Additive Inverse in C

$\forall\ (a + bi)\epsilon C\ \exists\ (x + yi)\epsilon C$ such that

$(a + bi) + (x + yi) = 0 + 0i$

11. Multiplicative Inverse in C

$\forall\ (a + bi)\epsilon C\ \exists\ (x + yi)\epsilon C$ such that

$(a + bi)(x + yi) = 1 + 0i$

Exercise 7-11-10

1. Represent $\frac{1}{a + bi}$ in the form $x + yi$.

2. Find the multiplicative inverse of each of the following complex numbers. Express results in $a + bi$ form.

a. $1 + i$
c. $3 - 3i$
e. $3 + 0i$
b. $6 - 2i$
d. $c - di$
f. $0 - 4i$

3. Perform the indicated operations. Express the results in the $a + bi$ form.

a. $\frac{1 + i}{1 - i}$
c. $\frac{1 - i}{1 + 3i} \cdot \frac{2 + 3i}{-1 + 4i}$
b. $\frac{6 + 3i}{2i}$
d. $\frac{3}{2 - 5i} - \frac{2i}{-2 - 5i}$

7-12 The solution of Quadratic Equations in C

A new field, the set of complex numbers and the operations defined on it, has been developed enabling us to find solution sets for any quadratic equation of the form $ax^2 + bx + c = 0$ where $a, b, c\epsilon R$, $a \neq 0$. Two examples, one completing the square, and one using the quadratic formula will be used to solve quadratic equations in C.

Example 7-12-1

Find the solution set of $x^2 - 4x + 5 = 0$ by completing the square.

$$x^2 - 4x + 5 = 0$$

$$x^2 - 4x = -5$$

$$x^2 - 4x + 4 = -5 + 4$$

$$(x - 2)^2 = -1$$

$$(x - 2)^2 + 1 = 0$$

$$(x - 2)^2 - (-1) = 0$$

$$(x - 2)^2 - (i)^2 = 0$$

$$(x - 2 - i)(x - 2 + i) = 0$$

We have proved that if $a \cdot b = 0$ then $a = 0$ or $b = 0$ for the reals. This is a theorem which holds for $a \varepsilon C$ and $b \varepsilon C$. You will be asked to prove this as an exercise.

Hence from above:

$x - 2 - i = 0$ or $x - 2 + i = 0$

$x = 2 + i$ $\quad$ $x = 2 - i$

The solution set is:

$$\{(2 + i), (2 - i)\}$$

Check:

$$x = 2 + i$$

$$(2 + i)^2 - 4(2 + i) + 5 \overset{?}{=} 0$$

$$4 + 4i + i^2 - 8 - 4i + 5 \overset{?}{=} 0$$

$$4 - 1 - 8 + 5 \overset{?}{=} 0$$

$$0 = 0$$

$$x = 2 - i$$

$$(2 - i)^2 - 4(2 - i) + 5 \overset{?}{=} 0$$

$$4 - 4i + i^2 - 8 + 4i + 5 \overset{?}{=} 0$$

$$4 - 1 - 8 + 5 \overset{?}{=} 0$$

$$0 = 0$$

Example 7-12-2

Find the solution set of the equation $x^2 - 4x + 5 = 0$ using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$$

$$x = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$x = \frac{4 \pm \sqrt{4(-1)}}{2}$$

$$x = \frac{4 \pm \sqrt{4i^2}}{2}$$

$$x = \frac{4 \pm 2i}{2} = 2 \pm i$$

The solution set is:

$$\{(2 + i), (2 - i)\}$$

Check as in Example 7-12-1

Exercise 7-12-3

Prove the following theorem:

V $a \epsilon C$, V $b \epsilon C$

If $a \cdot b = 0$ then $a = 0$ or $b = 0$,

7-13 Characteristics of Roots of the Quadratic Equation

The expression $b^2 - 4ac$ which appears under the radical sign of the quadratic formula is called the DISCRIMINANT of the quadratic equation.

The value, D, of $b^2 - 4ac$ for any quadratic equation with real coefficients will reveal certain characteristics of the roots of the given equation.

Exercise 7-13-1

1. Find the roots of each of the following equations. Then examine the discriminant of each equation. Complete the table by placing a (✓) in the columns which apply to each equation.

Equation I	Equation II	Equation III
$x^2 + 3x - 4 = 0$	$-x^2 + 6x - 9 = 0$	$x^2 - 10x + 28 = 0$

				Roots		Roots	
Equation	D>0	D<0	D=0	Real	Non-real	Equal	Unequal
I							
II							
III							

2. From the information in the above table write a generalization about characteristics of the roots of a quadratic equation when the value of the discriminant is positive, negative, zero.

Exercise Set 7-13-2

1. Without solving the equation, determine the nature of its roots, in each of the following:

a. $x^2 + 4x - 5 = 0$

b. $-5t^2 - t = 3$

c. $\sqrt{5}z^2 = 6z - 5$

d. $3x^2 = -7$

e. $x^2 + \frac{7}{4} = 7x$

f. $\sqrt{2}x^2 - x = 0$

2. For what value or values of k does $x^2 - 4x - k = 0$ have non-real solution?

3. For what value or values of p does $px^2 - 6x + p = 0$ have non-real solution?

Exercise Set 7-13-2 (cont.)

4. For what value or values of m does the graph of $y = x^2 - 3x + m$ cut the axis in two points.

5. For what value or values of m in the quadratic equation $mx \quad - 4x + 3 = 0$ is 2 an element of the solution set?

6. For what value or values of k is the graph of $y = x^2 - kx + k + 8$ tangent to the x-axis?

7. For what value or values of k is one solution of $x^2 - 6x + k = 0$ twice the other solution?

8. Flow chart the procedure for finding the solution set for any quadratic equation.

9. Write the program to print out the roots of any quadratic equation. Use your program to find the solution set of each of the following.

   a. $4x^2 + 2x + 1 = 0$

   b. $-x^2 + 7x - 3 = 0$

   c. $25x^2 - 3x + \frac{1}{4} = 0$

   d. $x^2 - 3x + 9 = 0$

   e. $-2x^2 + 6x + 9 = 0$

   f. $3x^2 + 3x + 1 = 0$

   g. $5x^2 - 6x + 2 = 0$

   h. $10x^2 - 29x + 10 = 0$

   i. $-x^2 + 2x + 5 = 0$

   j. $x^2 + 6 = 0$

   k. $.637x^2 + .035x + 3.716 - 0$

   l. $269x^2 - 63x + 3.671 = 0$

   m. $7615.1x^2 + 2613.7x = 2571.8$

n. $6.25x^2 = 1$

o. $x^2 + .0872x + 1 = 0$

p. $14.128x^2 + 6x + .637 = 0$

10. Consider the quadratic equation $\frac{a}{b}x^2 + \frac{c}{d}x + \frac{e}{f} = 0$ with rational number coefficients. That is, let a, b, c, d, e, f be integers and $b \neq 0$, $d \neq 0$, $f \neq 0$. This equation can be converted to an equation with integer coefficients. Write, a BASIC program that reads a, b, c, d, e, and f from a DATA statement, computes, and prints out the equivalent equation with integer coefficients, Then have the program compute and print out the solution set of the resulting equation.

7-14 Non-real Roots as Conjugate Pairs

Non-real roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, $a,b,c \varepsilon R$ exist in conjugate pairs. This can be shown to be the case by solving the quadratic equation by use of the quadratic formula.

We arrive at

$$x_1 = \frac{-b + i\sqrt{|D|}}{2a} \quad \text{and} \quad x_2 = \frac{-b - i\sqrt{|D|}}{2a}$$

The solution set is:

$$\left\{\frac{-b}{2a} + \frac{i\sqrt{|D|}}{2a}, \frac{-b}{2a} - \frac{i\sqrt{|D|}}{2a}\right\}$$

whose elements are obviously conjugates of each other.

Exercise 7-14-1

Explain why we placed the $|D|$ under the radical instead of D in the above expressions.

Polynomial Functions

Chapter 8

8-1 Introduction

In previous chapters we studied linear and quadratic functions. Both of these types of functions are sub-classes of a general class of functions known as polynomial functions. In this chapter we will analyze the properties and graphs of polynomial functions.

8-2 Definitions Related to Polynomial Functions

We will begin our study by considering a general polynomial expression, P.

Definition 8-2-1 Polynomial Expression

An algebraic expression P is a polynomial expression in one variable x if and only if P is of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0$$

where n is any non-negative integer and a_i, for $n \geq i \geq 0$, are elements of a given field, F, and x is a variable whose replacement set is F. We say that P is a polynomial over the field F.

The definition suggests that a polynomial may consist of many terms or simply one term.

Example 8-2-2

1. $2x + 3$
2. $3x^2 + 5x - 7$
3. 0
4. 5
5. $-\sqrt{7}x^3 + x^2 - 3x + \sqrt{2}$
6. $\frac{3}{4} x^5 - \frac{1}{2}$
7. $(3 + 4i)x^3 - (2 - 7i)x + 3i$
8. x^{13}

Each one of the expressions in Example 8-2-2 is a polynomial expression over a field of numbers. Name the field over which each of these polynomials may be defined? The number, a_i, in each term is referred to as the coefficient of that term. Polynomials, such as example 4 above, are called constant polynomial expressions; 0 is called the zero polynomial.

Definition 8-2-3 Degree of a Polynomial

The degree of a polynomial P is n where P is defined by

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$$

if and only if $a_n \neq 0$.

Carefully consider the characteristics of the polynomials from Example 8-2-2 which are listed in the following figure. After studying the information which is provided complete the missing parts of the chart.

	Polynomial	Degree	Coefficients (a_i, $n \geq i \geq 0$)
1	$2x + 3$	1	2, 3
2	$3x^2 + 5x - 7$	2	
3	0	none	0
4	5		5
5	$-\sqrt{7}x^3 + x^2 - 3x + \sqrt{2}$	3	
6	$\frac{3}{4}x^5 - \frac{1}{2}$		
7	$(3 + 4i)x^3 - (2 - 7i)x + 3i$		$3 + 4i, 0, -2 + 7i, 3i$
8	x^{13}		

Figure 8-2-4

Exercise 8-2-5

Write each of the following expressions in the form

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

if possible. Those expressions which cannot be written in this form are not polynomial expressions, explain why.

1. $3x^2 + 2(x)^{2/3} - 5$
2. $(3 + 2i)x^3 - (7 + 3i)x^2 + 2i$
3. $(3x + 2)(x^2 + 2x - 4)$
4. $2x^3 + \sqrt{x} + 3$
5. $2^x + 1$
6. $((((x + 2)x + 3)x - 7)x + 8)\ x + 4$
7. $4x^4 - 3x^3 - (3 + 2i)x + \sqrt{2}x$
8. $x^4 + 3x^2 - 2x + 3x^{-2}$

The text material from this point through Section 8-9 will deal with polynomials over the field of real numbers, R.

When evaluating polynomials over the real number field the principles of closure for addition and multiplication suggest that each real number replacement for the variable corresponds to a real number. This real number is the value of the polynomial.

Example 8-2-6

Consider the polynomial expression

$$Q: \quad x^2 + 3x - 10.$$

The values of the polynomial for x = 0, x = -10, and x = 2 are

$$Q(0) = (0)^2 + 3(0) - 10 = -10$$

$$Q(-10) = (-10)^2 + 3(-10) - 10 = 60$$

$$Q(2) = (2)^2 + 3(2) - 10 = 0$$

When the numerical replacement for the variable results in the value of the expression becomeing zero that number is called a zero of the polynomial. As in Example 8-2-6, Q(2) = 0, therefore 2 is called a zero of the polynomial expression Q. Since Q(-5) = 0, -5 is also a zero of Q.

The preceding polynomial expression can easily be evaluated for any replacement. Polynomials of higher degree would be more tedious.

Example 8-2-7

Evaluate the following polynomial, R, when x is -3.

$$R: \; x^6 - 5x^5 + 2x^3 - 7x^2 + 2x + 14$$

The traditional approach towards evaluation would be to find the value of

(1) $(-3)^6 - 5(-3)^5 + 2(-3)^3 - 7(-3)^2 + 2(-3) + 14.$

Complete the computation for R(-3) in (1) above.

The computation you did in the previous example proved to be quite tedious. Since evaluation of any polynomial expression, except for those of lower degree, is time consuming we will direct the computer to do this computation for us. While the following discussion will be unfamiliar, the algorithm which is developed will be a valuable aide for both machine and hand calculation.

Example 8-2-8

Evaluate the polynomial R: $x^6 - 5x^5 + 2x^3 - 7x^2 + 2x + 14$ when x is -3.

Factoring shows that R may be written in another form. Study the following steps carefully.

$$\begin{aligned} x^6 - 5x^5 + 2x^3 - 7x^2 + 2x + 14 &= 1x^6 - 5x^5 + 0x^4 + 2x^3 - 7x^2 + 2x + 14 \\ &= (1x^5 - 5x^4 + 0x^3 + 2x^2 - 7x + 2)x + 14 \\ &= ((1x^4 - 5x^3 + 0x^2 + 2x - 7)x + 2)x + 14 \\ &= (((1x^3 - 5x^2 + 0x + 2)x - 7)x + 2)x + 14 \\ &= ((((1x^2 - 5x + 0)x + 2)x - 7)x + 2)x + 14 \\ &= (((((1x - 5)x + 0)x + 2)x - 7)x + 2)x + 14 \\ &= ((((((1)x - 5)x + 0)x + 2)x - 7)x + 2)x + 14 \end{aligned}$$

With this factored form of R let us evaluate the polynomial at -3.

$$((((((1)(-3) - 5)(-3) + 0)(-3) + 2)(-3) - 7)(-3) + 2)(-3) + 14$$

$$((((((-3) - 5)(-3) + 0)(-3) + 2)(-3) - 7)(-3) + 2)(-3) + 14$$

$$(((((-8)(-3) + 0)(-3) + 2)(-3) - 7)(-3) + 2)(-3) + 14$$

$$(((((24) + 0)(-3) + 2)(-3) - 7)(-3) + 2)(-3) + 14$$

$$((((24)(-3) + 2)(-3) - 7)(-3) + 2)(-3) + 14$$

$$((((-72) + 2)(-3) - 7)(-3) + 2)(-3) + 14$$

$$(((-70)(-3) - 7)(-3) + 2)(-3) + 14$$

$$(((210) - 7)(-3) + 2)(-3) + 14$$

$$((203)(-3) + 2)(-3) + 14$$

$$((-609) + 2)(-3) + 14$$

$$(-607)(-3) + 14$$

$$(1821) + 14$$

$$1835$$

$\therefore$ The value of R when x is -3 is 1835

One would hardly consider the above evaluation an algorithm which is less work than the method shown in Example 8-2-7.

Consider the following flow chart, Figure 8-2-9. Trace through it using the polynomial R from the previous example. Complete table 8-2-10 as you trace this flow chart.

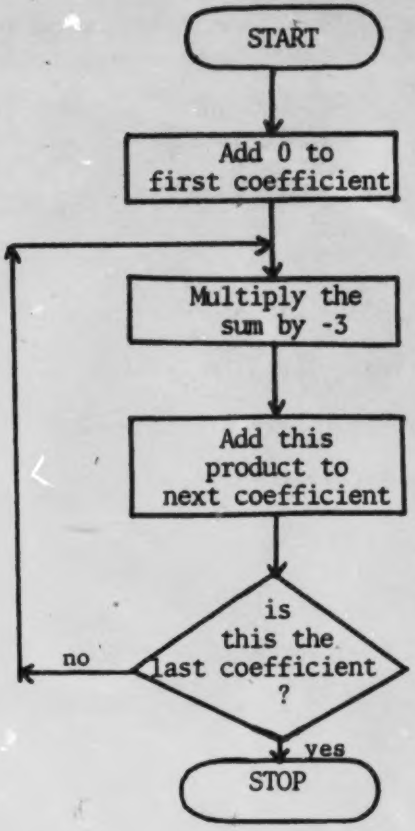

START
Add 0 to first coefficient
Multiply the sum by -3
Add this product to next coefficient
is this the last coefficient ?
no
yes
STOP

Figure 8-2-9

Coefficients:	1	-5	0	2	-7	2	14
Products:	0	-3	24				
Sum:	1	-8					

Table 8-2-10

This method for evaluating a polynomial is usually referred to as synthetic substitution. To illustrate a general pattern, consider the third degree polynomial expression, P.

$$a_3x^3 + a_2x^2 + a_1x^1 + a_0,\ a_3 \neq 0$$

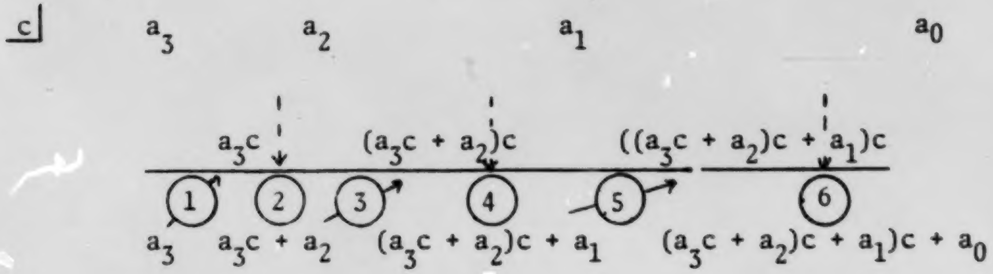
Study the following sequence of operations for evaluating P at c where c is any real number

1. Multiply a_3 by c	a_3c
2. Add a_2	$a_3c + a_2$
3. Multiply this sum by c.	$(a_3c + a_2)c$
4. Add a_1.	$(a_3c + a_2)c + a_1$
5. Multiply this sum by c.	$((a_3c + a_2)c + a_1)c$
6. Add a_0	$((a_3c + a_2)c + a_1)c + a_0$

The distributive property may be applied to the expression in step 6, above.

$$((a_3c + a_2)c + a_1)c + a_0 = a_3c^3 + a_2c^2 + a_1c + a_0$$

This shows that the expression in step 6 is the value of the polynomial, P, for any real number C. The six steps above are restated in abbreviated form below. The circled numeral indicates the sequence of operations from above which is being illustrated.



It should be clear that step 6 is the value of the polynomial expression P at any real number c.

Example 8-2-11

Evaluate the polynomial

$$Q: \ x^7 - 6x^4 + 3x^2 - 2x + 7$$

at $x = 4$

$$\begin{array}{r|rrrrrrrr} 4 & 1 & 0 & 0 & -6 & 0 & +3 & -2 & +7 \\ & & 4 & 16 & 64 & 232 & 928 & 3724 & 14888 \\ \hline & 1 & 4 & 16 & 58 & 232 & 931 & 3722 & 14895 \end{array}$$

. . $Q(4) = 3759$

Exercise 8-2-12

1. If R is the polynomial $4x^3 + 3x^2 + 1$ find its value at 0, -1, ½, 2, a - 1. Are any of these replacements zeros of R? Name them.

2. Find as many zeros as you can for each of the following polynomials:

   a. $x^2 + 2x - 8$

   b. $x^3 - 2x^2 + x - 2$

   c. $x^5 + 32$

   d. $x^4 - 16$

3. Expand the flow chart for synthetic substitution Figure 8-2-9 and from this flow chart write a program which will evaluate a polynomial expression for six specific elements from the replacement set. Your program should:

   a. Read from a DATA statement the degree, n, of the polynomial.

   b. Read from a DATA statement the coefficients A(n), A(n-1), A(n-2). . . ., A(1), A(0). Make provisions for a 25th degree polynomial.

   c. Read from a DATA statement one replacement for the variable, x.

   d. Compute the value of the expression for this replacement, P(x).

   e. Print the values of x and P(x).

   f. Make note if the replacement is a zero of the polynomial.

   g. Repeat steps c through g until all six values of the polynomial have been found.

4. Evaluate the following polynomials for the replacements 0, -2, 0.5, 1, -2.3, 4.0.

   1. F: $2x^6 - 5x^4 + 3x^3 - 7x^2 + 9x + 6$

   2. H: $2w^4 + 3w^2 - w + 1$

   3. P: $3.6x^4 - 0.00628x^2 - 149.36$

   4. G: $16.354x^3 - 3.14x^2 + .024x - .0036$

   5. F: $19.3865r^6 - 12.1r^5 - 1.852r^4 + (15/3)r^3 + 4r^2 + 2$

8.3 Graphing Polynomial Functions

In the previous section we observed that for a polynomial expression over the field of real numbers, the expression represented a unique real number for each real replacement for the variable. This idea should suggest that every polynomial over the field R defines a function.

Definition 8-3-1 Polynomial Function

The set of ordered pairs determined by $y = P(x)$ is a polynomial function if and only if $P(x)$ is a polynomial expression.

The emphasis in this section will be on graphs of polynomial functions. This emphasis will not be how to graph the function itself but how to interpret the graph of the polynomial function in a meaningful way. After studying some of the important characteristics of the graphs of these functions we will, in later sections, develop algebraic methods for finding this information.

Let us begin with a practical problem which requires analysis of the graph of a polynomial function.

Example 8-3-2

You are given 1000 feet of fencing material. If you desire to fence the greatest area possible in a rectangular plot of ground what will be the dimensions of this plot?

Given: 1000 feet of fencing material

Constraint: fence a rectangular region

Objective: maximize area enclosed by this fence

From this information we may form two equations in length (l) and width (w):

$$(1) \quad 2l + 2w = 1000$$

$$(2) \quad lw = A$$

Since we want to consider area we must use equation (2) and, by substitution, form a function of area in terms of length or width.

$$A = lw \text{ and } 2l + 2w = 1000$$

$$A = lw \text{ and } l = 500 - w$$

$$(A = (500 - w)w) \rightarrow (A = -w^2 + 500w)$$

Now $-w^2 + 500w$ is a polynomial expression, therefore, $A = -w^2 + 500w$ defines a polynomial function. Some sample points from this function may be seen below.

{(0, 0), (1, 499), (2, 996), (4, 1984), (100, 40,000), (200, 60000), (300, 60000), (400, 40000), (500, 0), ...}

The graph appears in Figure 8-3-3.

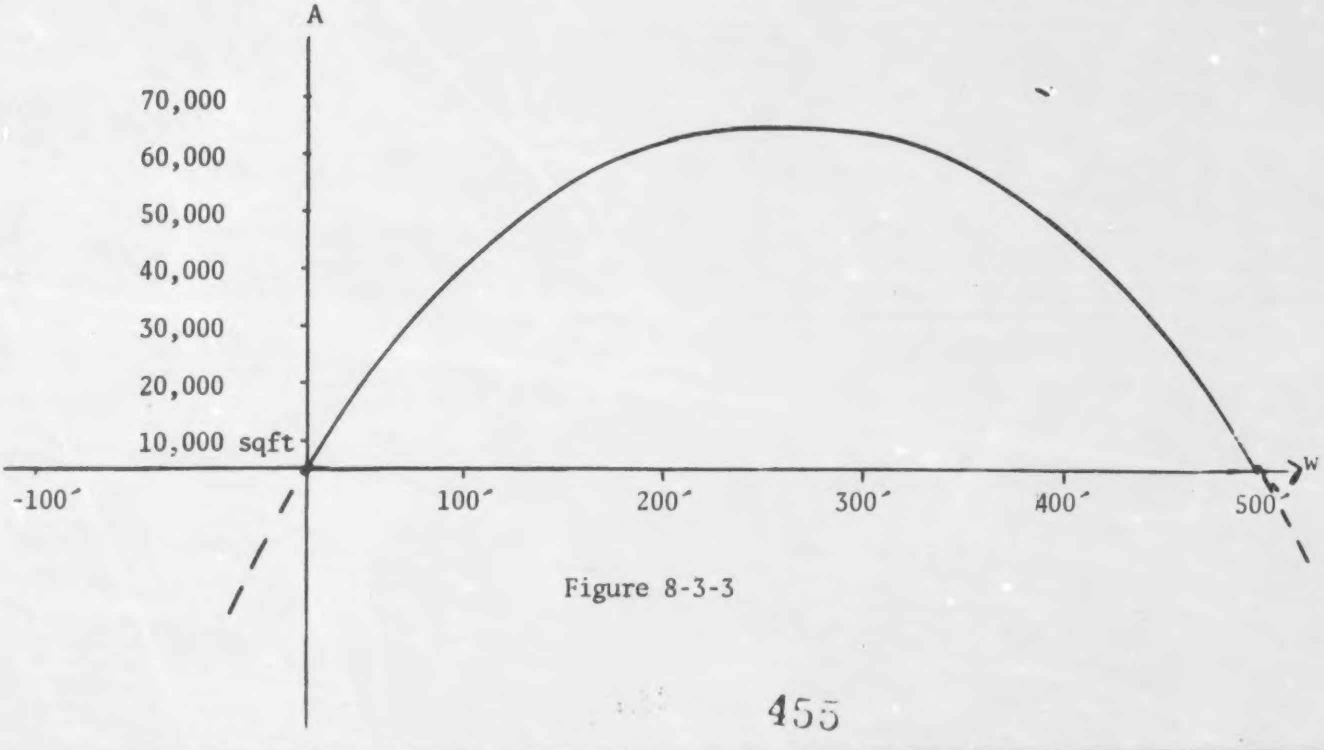


Figure 8-3-3

Although the graph of $A(w) = -w^2 + 500w$ is shown in Figure 8-3-3 in its entirety for the problem we are considering we shall consider only w for which $0 < w < 500$. Why?

> The maximum area for our fenced plot is the highest point on the graph, which appears to be when the width is 250 feet. When w = 250 ft. 1 = 250 ft. so it appears that a square 250 ft. per side yields maximum area.

Notice that the zeros of the polynomial are those widths for which $A = 0$. In general, the zeros of a polynomial function, $y = P(x)$, are those values of x for which $y = 0$.

The domain of the function, $\{(w,A) | A = -w^2 + 500w\}$, is the set of real numbers, R and its range is subset of R. In general, the domain of a polynomial function over R is R and the range is either R or a subset of R. Can you name a polynomial function over R whose range is R?

A final note about polynomial functions in general. All polynomial functions over the field, R, are continuous functions.

Exercise 8-3-4

1. Write a program to illustrate the maximum value of the function defined in Example 8-3-2 using only:
   a) replacements which differ by a integer
   b) replacements which differ by $\frac{1}{10}$ of a unit
   c) replacements which differ by $\frac{1}{100}$ of a unit

*2. Argue albegraically that 250 feet is the width for which maximum area is obtained.

In problems 3-4, find a polynomial function which meets the specifications outlined in each problem. Use the computer to obtain enough ordered pairs to graph this function. Devise a program causing the computer to find the information the problem asks for. Your answer should be accurate to a tolerance of .0001.

3. You are given 1000 feet of fencing material. Find the dimensions of a rectangular plot of ground with maximum area if you may use an existing fence for one side.

4. An open rectangular box is to be made from a piece of tin 27 inches square by cutting a square from each corner and folding up the sides. Find the dimensions of the box of largest volume.

5. For the polynomial function defined by $y = x^4 - x^3 - 13x^2 - x + 112$ find the point at which the graph of the function is closest to the x-axis.

In order to become more familiar with the behavior of the graphs of higher degree polynomial functions, graph the functions in Exercise 6-10 below.

6. $\{(x,y)|y = x^3 - 3x^2 - 4x + 12\}$
7. $\{(x,y)|y = -x^3 + 3x^2 + x - 3\}$
8. $\{(x,y)|y = x^4 - x^3 - 13x^2 - x + 12\}$
9. $\{(x,y)|y = -2x^5 + 5x^4 + 12x^3 - 23x^2 - 16x + 12\}$
10. $\{(x,y)|y = 1.2765x^5 - 3.14159x^4 - 428x^3 + 15x^2 + 0.0000456x^2 - 16x + 12.75\}$
11. What is the domain and range of each function defined in Problem 6-10 above?

8-4 Algebra of Polynomials

Let us consider a mathematical system consisting of the set of all polynomials in one variable over the field of real numbers. In this system we will deal with two operations, addition and multiplication. The objective of this study is to determine if this system is an additive group, a multiplicative group and/or a field.

Consider the following polynomial, P,

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$$

Let us agree that P is a polynomial over the field of real numbers. We already know that for any real number replacement for x, each term, a_ix^i, $n \geq i \geq 0$, is a real number. This is due to the closure property for multiplication. The closure property for addition guarantees that the sum of these terms is also a real number. Since these polynomial expressions over the field of real numbers represent real numbers and the terms of these polynomials represent real numbers, the field properties of the real number set suggest how to compute sums and products of polynomials.

Example 8-4-1

Consider the polynomials

$$P:\ 3x^2 + x + 6$$

$$Q:\ 15x^3 - 3x + 1$$

a. Find the sum of P + Q.

$$\begin{aligned} P + Q &= (3x^2 + x + 6) + (15x^3 - 3x + 1) \\ &= (0 + 15)x^3 + (3 + 0)x^2 + (1 - 3)x + (6 + 1) \\ &= 15x^3 + 3x^2 - 2x + 7 \end{aligned}$$

b. Find the product of P and Q.

$$\begin{aligned} P \cdot Q &= (3x^2 + x + 6)(15x^3 - 3x + 1) \\ &= (3x^2 + x + 6)15x^3 + (3x^2 + x + 6)(-3x) + (3x^2 + x + 6)(1) \\ &= 45x^5 + 15x^4 + 90x^3 - 9x^3 - 3x^2 - 18x + 3x^2 + x + 6 \\ &= 45x^5 + 15x^4 + (90 - 9)x^3 + (-3 + 3)x^2 + (-18 + 1)x + 6 \\ &= 45x^5 + 15x^4 + 81x^3 - 17x + 6 \end{aligned}$$

Exercise 8-4-2

In problems 1-4, the first polynomial is P and the second is Q. Find, for each problem P + Q, P - Q and P · Q.

1. $4x^3 + 3x^2 - x + 5;\ 5x^3 - 3x^2 + 2x - 1$
2. $7x^4 - 4x^3 + 2x^2 + 6;\ x^3 + 2x^2 + 2x - 2$
3. $x + 9 - x^3;\ 3 - 2x + 3x^2$
4. $-2x^6 - 3x^3 + x - 4;\ 2x^4 - x^3 + 2x - 2$

From the above exercise, it seems that the sum of any two polynomials is another polynomial. If one considers the zero polynomial, 0, then the system of polynomial expressions has an additive identity. If $P(x)$ represents any polynomial, $-P(x)$ is also a polynomial such that $P(x) + (-P(x)) = 0$.

Example 8-4-3

If $P(x) = 5x^7 + 3x^6 - 4x^5 - x^3 + 15x + 2$

then $-P(x) = -(5x^7 + 3x^6 - 4x^5 - x^3 + 15x + 2)$

or $-P(x) = -5x^7 - 3x^6 + 4x^5 + x^3 - 15x - 2$

and $P(x) + -(P(x)) = 0$

Since a polynomial expression represents a real number for a real number replacement for the variable it follows that polynomial expressions are associative with respect to addition.

We may conclude that the system of polynomial expressions with the operation of addition forms a group. This is sometimes said to be an additive group.

Exercise 8-4-4

1. Given these polynomials

   P: $3x^2 - 7x + 2$

   Q: $5x^3 + 4x - 3$

   R: $x^2 + 4x - 1$

   Verify:

   a) $P + Q = Q + P$

   b) $(P + Q) + R = P + (Q + R)$

   c) $P \cdot Q = Q \cdot P$

   d) $(P \cdot Q)R = P \cdot (Q \cdot R)$

   e) $P \cdot (Q + R) = P \cdot Q + P \cdot R$

2. Name the polynomial expression which is the multiplicative identity of this system of polynomials.

The system of polynomial expressions along with the operation of multiplication is closed and multiplication is associative. It is also true that 1 is the multiplicative identity. However, the multiplicative inverse of a polynomial is not, in general, a polynomial.

Example 8-4-5

Find the multiplicative inverse of x + 1.

Assume, P(x) is a polynomial over the field of real numbers for which

$$(x + 1)\ P(x) = 1$$

$$\text{or} \qquad P(x) = \frac{1}{x + 1}$$

But, $\frac{1}{x + 1}$ is not a polynomial expression.

This means that the set of all polynomials over the field R is not a group with respect to multiplication.

We can conclude from the above discussion that the set of polynomial expressions along with the operations of addition and multiplication does not form a field. However, this system satisfies the same field properties that the system of integers satisfies. That is, the system is an additive group, it is closed with respect to multiplication, there exists a multiplicative identity element, the system is associative for multiplication and distributive with respect to multiplication over addition. Any system possessing these properties of the integers is called an integral domain. This system of polynomials over the field of real numbers is an integral domain.

8-5 The Division Process

The systems of integers and polynomials are also similar when the process of division is considered. We have noted that a quotient of two polynomials need not be a polynomial, just as a quotient of two integers need not be an integer. However, it is possible to divide one integer by another, obtaining a quotient and a remainder, each of which is an integer. For example, 354 divided by 13 gives a quotient of 27 and a remainder of 3, as shown below.

$$\begin{array}{r} 27 \\ 13\overline{)354} \\ \underline{26} \\ 94 \\ \underline{91} \\ 3 \end{array} \qquad \text{that is } 354 = 27 \times 13 + 3$$

In the same way, it is possible to divide one polynomial by another, obtaining a quotient and a remainder each of which is a polynomial.

Example 8-5-1

Divide $6x^3 - 8x^2 + 5x - 1$ by $3x^2 - x + 2$

The procedure and reasoning of this division process is essentially the same as that of long division of real numbers.

$$
\begin{array}{rrl}
 & 2x - 2 & \\
3x^2 - x + 2 & \overline{)6x^3 - 8x^2 + 5x - 1} & \\
 & \underline{6x^3 - 2x^2 + 4x \phantom{{}-1}} & = 2x(3x^2 - x + 2) \\
 & -6x^2 + x - 1 & \\
 & \underline{-6x^2 + 2x - 4} & = -2(3x^2 - x + 2) \\
 & -x + 3 &
\end{array}
$$

Expressing this result in the form of an equation:

$$6x^3 - 8x^2 + 5x - 1 = (2x - 2)(3x^2 - x + 2) + (-x + 3)$$

Where $2x - 2$ is the quotient

and $-x + 3$ is the remainder.

The quotient and remainder obtained when one integer is divided by another are unique. Symbolically, this means that if a and b are integers, with $b > 0$, then there is one, and only one, integer q and one and only one integer r such that:

$$a = qb + r$$

and

$$0 \leq r < b$$

The integer q is the quotient and r is the remainder when a is divided by b.

In a similar manner, the quotient and remainder obtained by dividing one polynomial by another are unique. We shall state this fact as a theorem but will not provide a proof for it.

Theorem 8-5-2

Let $A(x)$ and $B(x)$ be polynomials, $B(x) \neq 0$. There are polynomials $Q(x)$ and $R(x)$ where $R(x)$ is either the zero polynomial or is of lower degree than $B(x)$, such that

$$A(x) = Q(x) \cdot B(x) + R(x).$$

Example 8-5-3

Find the quotient and remainder upon dividing the polynomial $3x^4 - 7x^2 + 2x$ by the polynomial $x^2 + x - 3$.

$$
\begin{array}{rrrrrr}
 & 3x^2 & -\,3x & +\,5 & & \\
\cline{2-6}
x^2 + x - 3\big) & 3x^4 & +\,0x^3 & -\,7x^2 & +\,2x & +\,0 \\
 & 3x^4 & +\,3x^3 & -\,9x^2 & & \\
\cline{2-6}
 & & -3x^3 & +\,2x^2 & +\,2x & \\
 & & -3x^3 & -\,3x^2 & +\,9x & \\
\cline{3-6}
 & & & 5x^2 & -\,7x & \\
 & & & 5x^2 & +\,5x & -\,15 \\
\cline{4-6}
 & & & & -12x & +\,15
\end{array}
$$

According to the long division above, $3x^2 - 3x + 5$ is the quotient and $-12x + 15$ is the remainder when $3x^4 - 7x^2 + 2x$ is divided by $x^2 + x - 3$. Note that we provided zeros for the missing term of degree three and also for the missing constant term. Expressing this result as an equation suggested by Theorem 8-5-2 we have:

$$3x^4 - 7x^2 + 2x = (3x^2 - 3x + 5)(x^2 + x - 3) + (-12x + 15)$$

Exercise 8-5-4

Find the quotient and remainder determined by each of the following divisions. Write your results in the form $A(x) = Q(x) \cdot B(x) + R(x)$.

1. $2x^3 - 3x^2 + 6x^2 - 2 \div x - 4$
2. $x^5 - 4x^3 + 5x^2 - 5 \div x + 1$
3. $2x + 4 \div x - 2$
4. $2x^4 - x^3 + 7x^2 - 2x - 2 \div 2x^2 - x + 1$
5. $3x^5 - 2x^4 + x^3 - 15x^2 + 10x - 5 \div 3x^2 - 2x + 1$
6. $x^4 + x^2 - 1 \div x^2 + x + 1$
7. $3x^3 - x + 1 \div 2x^2 - 1$
8. $x^5 - 4x^3 + 5x^2 - 5 \div x + 1$
9. Check to see that in each problem above, 1-8, the remainder, $R(x)$ is either the zero polynomial or a polynomial of degree less than the degree of the divisor.

10. The above exercises illustrate examples of Theorem 8-5-2. What generalization does this theorem suggest about the remainder $R(x)$ when dividing a polynomial of degree greater than zero by a first degree polynomial.

11. For each of the following polynomial pairs, first divide $A(x)$ by $B(x)$ where $B(x)$ is of the form $x - r$. Compare this quotient with the results obtained by evaluating $A(x)$ at r, using synthetic substitution.

	$A(x)$	$B(x)$
a)	$x^3 + 3x^2 + 3x + 1$	$x + 1$
b)	$2x^3 - 3x^2 + 6x - 2$	$x - 4$
c)	$3x^3 + 6x - 2$	$x + 2$
d)	$2x^4 - x^3 + 7x^2 - 2x - 2$	$x - \frac{1}{2}$
e)	$x^8 - 1$	$x - 2$

*12. Explain, using a specific example, why synthetic substitution of r, or synthetic division as it is called in this instance, gives equivalent results as that of dividing by the polynomial $x - r$.

13. Prove that if $(Ax) = Q(x) \cdot (x - r) + R(x)$ then $A(r) = R(x)$ where $R(x)$ is a real number.

14. Discuss what conclusions may be established about r if $A(x) = Q(x) \cdot (x - r) + 0$. About $(x - r)$?

8-6 Factoring Polynomial Expressions

Let us begin this section by reviewing some of the conclusions you were directed to reach in Exercise 8-5-4. As a basis for later discussions let us first consider the results of exercise number 10.

If a polynomial, $P(x)$ degree $P(x) > 0$, is divided by a first degree polynomial what generalization about the remainder, $R(x)$, does Theorem 8-5-2 suggest ?

1. Theorem 8-5-2 tells us that $P(x) = Q(x)D(x) + R(x)$ where degree $R(x) <$ degree $D(x)$ or $R(x)$ is the zero polynomial.

2. Since $P(x)$ is being divided by a first degree polynomial, we may express $D(x)$ as $x - a$. Thus we have

$$P(x) = Q(x)\ (x - a) + R(x)$$

and the degree of $R(x)$ is less than one or $R(x) = 0$.

3. To say that the degree of $R(x)$ is less than one or $R(x) = 0$ is equivalent to saying that $R(x)$ is a constant polynomial or $R(x) = 0$.

Hence, the conclusion to the exercise would be that $R(x)$ is a real number.

With this background we may consider problems 13 and 14 of Exercise 8-5-4 as a Theorem.

Theorem 8-6-1 The Remainder Theorem

If b is the remainder when $P(x)$ is divided by $x - a$, then $P(a) = b$.

proof: If b is the remainder when $P(x)$ is divided by $x - a$ then

$$
\begin{array}{rl}
 & P(x) = Q(x)\ (x - a) + b \\
\text{now} & P(a) = Q(a)\ (a - a) + b \\
\text{or} & P(a) = Q(a) \cdot 0 + b \\
\text{and} & P(a) = b
\end{array}
$$

Exercise 14 is nothing more than a question about a specific instance of Theorem 8-6-1. That is, if

$$P(x) = Q(x)(x - a) + b \text{ and } b = 0$$

what is true about a and $x - a$. The correct conclusion may be read from the following theorem.

Theorem 8-6-2 The Factor Theorem

If $P(x)$ is a polynomial of degree greater than zero, r is a zero of $P(x)$ if and only if $x - r$ is a factor of $P(x)$.

Before we consider additional methods for determining the factors of a polynomial let us look at what a finite subset of a polynomial function tells us about the zeros of that function.

Consider the function $P(x) = x^3 - 3x + 2$ a subset of the ordered pairs of P(x) may be seen below.

x	P(x)
-10	-968
-5	-108
0	2
5	112
10	972

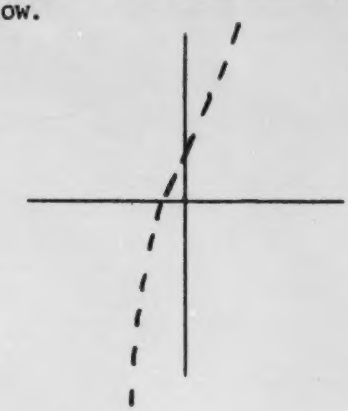


Figure 8-6-3

The 5 element subset of the function P(x) shown in Figure 8-6-3 suggests that there is a zero in the interval $-5 < x < 0$ because $P(-5) < 0$ and $P(0) > 0$. One might hastily conclude that the function P(x) is a strictly increasing function. Looking at a more extensive subset of the function P(x), Figure 8-6-4, we see that indeed there was a zero in the interval $-5 < x < 0$, it was -2. Again, the conclusion that P(x) was strictly increasing is upheld.

x	P(x)
-10	-968
-8	-486
-6	-196
-4	-50
-2	0
0	2
2	4
4	54
6	200
8	490
10	972

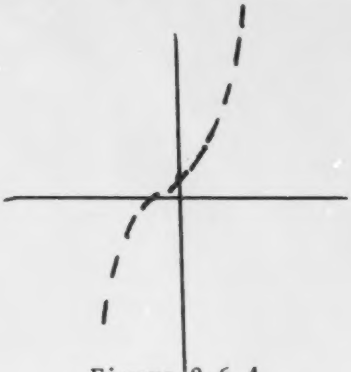


Figure 8-6-4

However, considering the interval from -2 to 2, Figure 8-6-5, we find our original conclusion about the zeros of this function was incomplete. In addition, we see that the conclusion regarding the increasing nature of this function was indeed hasty.

x	P(x)
-2	0
-1	4
0	2
1	0
2	4

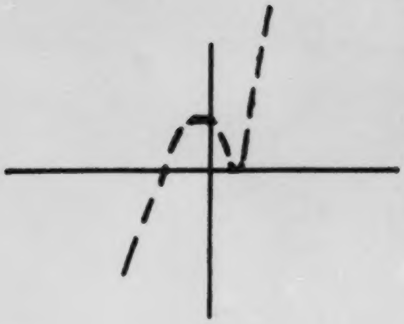


Figure 8-6-5

Now that we know that two of the zeros of P(x) are -2 and 1, the factors of the polynomial should be easily determined.

$P(x) = x^3 - 3x + 2$ and -2 is a zero

means $P(x) = Q(x) \cdot (x + 2)$.

To find Q(x) we divide P(x) by (x + 2):

$$\begin{array}{rrrrr} -2 & +1 & +0 & -3 & +2 \\ & & -2 & +4 & -2 \\ \cline{2-5} & +1 & -2 & +1 & 0 \end{array}$$

Hence $Q(x) = x^2 - 2x + 1$

and $P(x) = (x^2 - 2x + 1)(x + 2)$

or $P(x) = (x - 1)(x - 1)(x + 2)$

Now that P(x) is in its factored form and we have a rough idea of the nature of the graph of this function, there are many general characteristics of polynomials which we may discover from this example.

First, the polynomial P(x) is said to have three linear factors, $x - 1$, $x - 1$, and $x + 2$. It has two zeros, -2 is called a simple zero while, + 1 is said to be a multiple zero. In this instance +1 is zero of multiplicity 2.

Furthermore, we may investigate what happens to the functional value between the zeros of $P(x) = x^3 - 2x + 1$. To do this consider the following subset of the function P(x), Figure 8-6-6.

x	P(x)
-4	-50
-3.75	-39.4844
-3.5	-30.375
-3.25	-22.5781
-3	-16
-2.75	-10.5469
-2.5	-6.125
-2.25	-2.64062
-2	0
-1.75	1.89062
-1.5	3.125
-1.25	3.79687
-1	4
-.75	3.82812
-.5	3.375
-.25	2.73437
0	2
.25	1.26562
.5	.625
.75	.171875
1	0
1.25	.203125
1.5	.875
1.75	210937
2	4
2.25	6.64062
2.5	10.125
2.75	14.5469
3	20

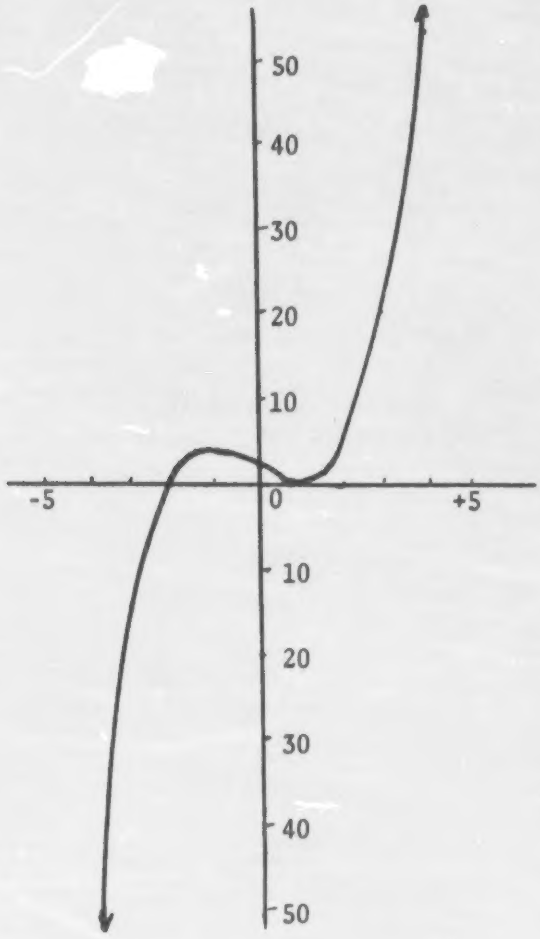


Figure 8-6-6

You probably have concluded that when a function has a simple, real zero the functional values on either side of this zero are opposite in sign. However, when a function contains a zero of multiplicity two, we find the functional values on either side of the zero to be the same sign. Use the following exercise to convince yourself of the apparent validity of these conclusions and project more generalizations about zeros of greater multiplicity than 2.

Exercise 8-6-7

Below you will find five polynomial functions. Beside each, you will find a finite subset of the corresponding function. Use division of polynomials to completely factor the function. Check to see that the zeros you find agree with the zeros of the function's subset. Describe the multiplicity of the zeros of each function. Generalize what happens to the functional values, f(x), as we choose values for x in an interval containing a zero of the function. Such a generalization should be made for simple zeros and zeros whose multiplicity is 2, 3, or 4.

1. $P(x) = x^2 - 10x + 21$

x	P(x)
-10	221
-9	192
-8	165
-7	140
-6	117
-5	96
-4	77
-3	60
-2	45
-1	32
0	21
1	12
2	5
3	0
4	-3
5	-4
6	-3
7	0
8	5
9	12
10	21

2. $Q(x) = -x^3 - 3x^2 + 10x + 24$

x	Q(x)
-10	624
-9	420
-8	264
-7	150
-6	72
-5	24
-4	0
-3	-6
-2	0
-1	12
0	24
1	30
2	24
3	0
4	-48
5	-126
6	-240
7	-396
8	-600
9	-858
10	-1176

3. $R(x) = x^4 - 8x^2 + 16$

x	R(x)
-10	9216
-9	5929
-8	3600
-7	2025
-6	1024
-5	441
-4	144
-3	25
-2	0
-1	9
0	16
1	9
2	0
3	25
4	144
5	441
6	1024
7	2025
8	3600
9	5929
10	9216

4. $S(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$

x	S(x)
-10	5632
-9	3430
-8	1944
-7	1000
-6	448
-5	162
-4	40
-3	4
-2	0
-1	-2
0	-8
1	0
2	64
3	250
4	648
5	1372
6	2560
7	4374
8	7000
9	10648
10	15552

5. $T(x) = x^6 + 4x^3 + 6x^2 + 4x + 1$

x	T(x)
-10	996561.
-9	528976.
-8	260449.
-7	116544.
-6	45985.
-5	15256
-4	3921
-3	664
-2	49
-1	0
0	1
1	16
2	129
3	904
4	4465
5	16296
6	47761.
7	119344.
8	264609.
9	534880.
10	1.00464E+06

6. Consider that $\{(a, P(a)), (b, P(b)), (c, P(c))\} \subset \{(x, P(x)) | x \varepsilon R\}$ for some polynomial $P(x)$. If the points $(a,P(a))$, $(b, P(b))$ and $(c, P(c))$ are graphically located as shown in Figure 8-9-8, answer the following questions.

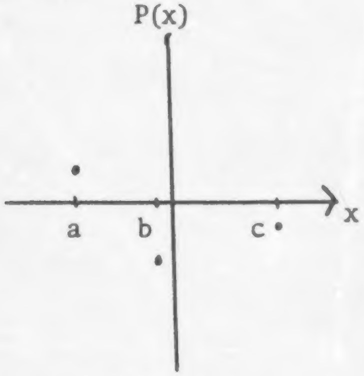


Figure 8-6-8

a) What is the minimum degree of the polynomial $P(x)$?

b) What generalization may be made about the number of real zeros, x, such that $a < x < b$?

c) What generalization may be made about the number of real zeros, x, such that $b < x < c$?

d) What generalization may be made about the number of real zeros, x, such that $x < a$ or $x > c$.

7. Given that 1, -1, and -2 are zeros of a polynomial:

a) find a third degree polynomial in x with these zeros.

b) How many third degree polynomials in x can be constructed with 1, -1, -2 as zeros and 1 as the leading coefficient?

c) How many fourth degree polynomials in x can be constructed with 1, -1, -2 as zeros and 1 as the leading coefficient?

8) Given the following sets of zeros construct the lowest degree polynomial possible.

a) $\{2, 3\}$	b) $\{-7, 11\}$
c) $\{\sqrt{5}, -\sqrt{5}\}$	d) $\{2 - 3\sqrt{2}, 2 + 3\sqrt{2}\}$
e) $\{2, 3 - \sqrt{5}\}$	f) $\{2 + 3i, 2 - 3i\}$
g) $\{-3 + \sqrt{5}i, -3 - \sqrt{5}i, 2\}$	h) $\{2 - \sqrt{3}i, -1\}$

9) Find the zeros for the following first and second degree polynomials.

a) $x - 7$

b) $\sqrt{3}x + 1$

c) $x^2 + 4x + 1$

d) $x^2 + 3x + 5$

e) $x^2 - (2\sqrt{2} + \sqrt{7})x + 2\sqrt{14}$

f) $x^2 + ix + 2$

8-7 Polynomials over the Complex Field.

It should be evident that all polynomials over the field of real numbers do not have zeros in that field. Consider the polynomial $P(x) = x^2 + 1$.

Example 8-7-1

Find all zeros of the polynomial $P(x) = x^2 + 1$.

Since the coefficients of the polynomial $x^2 + 1$ are 1, 0 and 1, this polynomial could be considered as a polynomial over the field of real numbers. To find the zeros of this polynomial we want to find those replacements for x from the set of real numbers, for which $x^2 + 1 = 0$. Now we know that i and -i satisfy this equation which means that i and -i are the zeros of $P(x) = x^2 + 1$. But, i and -i are not real numbers. Therefore, $P(x) = x^2 + 1$ has no zeros in the field of real numbers.

Since the coefficients of $x^2 + 1$ are also elements of the complex field, we could consider $P(x) = x^2 + 1$ as a polynomial over the complex field. If this is done the zeros i and $-i$ are within the defined field.

At this time we will accept, without proof, a theorem which deals with the zeros of any polynomial of degree greater than zero.

Theorem 8-7-2 The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial over the field of complex numbers and $P(x)$ is of degree greater than zero then $P(x)$ has at least one zero.

In more familiar terms, this theorem tells us that any polynomial, $P(x)$ of degree n, $n > 0$, may be written:

$$P(x) = Q_1(x)(x - r_1)$$

where $P(x)$ is a polynomial over the complex field, r_1 is a complex zero, and $Q_1(x)$ is a polynomial of degree $n-1$.

If we now consider $Q_1(x)$ in the sentence above, $P(x) = Q_1(x)\ (x - r_1)$, we will find that $Q_1(x)$ is a polynomial over the complex field. If the degree of $Q_1(x)$, that is, $n - 1$, is greater than zero, we could then write $Q_1(x) = Q_2(x)\ (x - r_2)$ and:

$$P(x) = Q_2(x)\ (x - r_2)\ (x - r_1) \text{ by substitution}$$

If the degree of $Q_2(x)$, $n - 2$, is greater than 0 we have:

$$P(x) = Q_3(x)\ (x - r_3)\ (x - r_2)\ (x - r_1)$$

Continuing these steps until the degree of $Q_n(x)$ is 1 we see that any nth degree polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

may be written as

$$P(x) = a_n(x - r_1)(x - r_2)\ .\ .\ .\ (x - r_n).$$

This conclusion was reached through repeated application of the Fundamental Theorem of Algebra.

Theorem 8-7-3

If $P(x)$ is a polynomial over the field of complex numbers and $P(x)$ is of degree greater than zero then $P(x)$ has exactly n zeros, not necessarily distinct, in the complex field.

Example 8-7-4

Find the zeros of the polynomial $P(x) = x^3 - 1$.

By observation, $P(1) = (1)^3 - 1 = 0$ which means that 1 is a zero of the polynomial.

We know, $x^3 - 1 = Q(x)\ (x - 1)$, by Theorem 8-10-2.

To find $Q(x)$ we divide $P(x)$ by $x - 1$.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

Therefore, $x^3 - 1 = (x^2 + x + 1)(x - 1)$

Now, $x^2 + x + 1 = x^2 + x + \frac{1}{4} + 1 - \frac{1}{4}$

$$= (x + \tfrac{1}{2})^2 + \frac{3}{4}$$

Since $\frac{3}{4} = -\frac{3}{4} i^2$

Substitution yields

$$x^2 + x + 1 = (x + \tfrac{1}{2})^2 - \frac{3}{4} i^2$$

$$= \left[(x + \tfrac{1}{2}) - \frac{\sqrt{3}}{2} i\right]\left[(x + \tfrac{1}{2}) + \frac{\sqrt{3}}{2} i\right]$$

$$= \left[x + (\tfrac{1}{2} - \frac{\sqrt{3}}{2} i)\right]\left[x + (\tfrac{1}{2} + \frac{\sqrt{3}}{2} i)\right]$$

which means that

$$x^3 - 1 = \left[x - (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)\right]\left[x - (-\frac{1}{2} - \frac{\sqrt{3}}{2} i)\right] (x - 1)$$

So the zeros of $x^3 - 1$ are $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$ and 1.

Exercise 8-7-5

1. Find all zeros of the polynomial $P(x) = x^3 - 7x + 6$ given -3 is one of the zeros.
2. Find all zeros of the polynomial $Q(x) = x^3 - 7x^2 + 13x - 6$ given $(x - 2)$ is one of its factors.
3. Find all factors of the polynomial $S(x) = x^4 + 15x^3 + 63x^2 + 23x - 102$ given zeros of -2 and 1.
4. Find all zeros of the polynomial $P(x) = x^3 - (3 + i)x^2 - (10 - 3i)x + 10i$ given $x - i$ is a factor.
5. Find all zeros of the polynomial $T(x) = x^4 - 4x^3 + 9x^2 - 8x + 14$ if $2 + \sqrt{3i}$ and $-\sqrt{2i}$ are two zeros of $T(x)$
6. Find a polynomial with complex coefficients and lowest possible degree if this polynomial has factors of $(x - 2)$ and $(x - 3 + 2i)$.
7. Find a polynomial with real coefficients and lowest possible degree if this polynomial has zeros of 2 and $3 + 2i$.

8-8 Polynomials over the Real Field.

The previous section should have led to the conclusion that: A polynomial of degree n over the field of complex numbers has complex zeros found in conjugate pairs if and only if the coefficients of the polynomial are real numbers. This proof of this theorem is not difficult and may be found in a number of second year algebra texts. This proof will not be dealtwith directly because our interest is in discovering the nature of the zeros of a polynomial.

A second theorem which relates to polynomials over the real number field is one which deals with the number of positive and negative zeros a given polynomial with real coefficient contains.

Theorem 8-8-1 Descartes' Rule of Signs

Let $P(x)$ be a polynomial with real coefficients. The number of positive zeros of $P(x)$ is either equal to the number of variations in sign of the polynomial coefficients or differs from this number of variations by a positive even integer. A zero of multiplicity a is counted a times. If the coefficients represent one variation in sign there is exactly one positive zero.

The number of negative zeros of $P(x)$ may be determined by the same evaluation of $P(-x)$.

Example 8-8-2

Determine the nature of the zeros of $P(x)$ if $P(x) = x^3 - 1$.

$P(x) = x^3 - 1$, which yields exactly one variation in sign. Therefore, there is exactly one positive zero.

$P(-x) = (-x)^3 - 1 = -x - 1$, which yields no sign variation. Therefore, there are no negative zeros.

Hence, $P(x)$ has one positive zero, no negative zeros, no zeros of 0, which means there are two complex zeros. The complex zeros are conjugate pairs. Why?

The student should verify that this evaluation is correct. Hint: one zero is +1.

Example 8-8-3

Determine the nature of the zeros of $R(x)$ if

$$R(x) = x^5 - 4x^4 - 3x^3 + 2x^2 - x + 7$$

$P(x)$ yields 4 sign changes. Therefore, there are 4, 2, or 0 positive zeros.

$$P(-x) = (-x)^5 - 4(-x)^4 - 3(-x)^3 + 2(-x)^2 - (-x) + 7$$

$$P(-x) = -x^5 - 4x^4 + 3x^3 + 2x^2 + x + 7$$

$P(-x)$ yields 1 sign change. Therefore, there is exactly one negative zero.

Hence, there are three possible cases:

	Positive Zeros	Zeros of Zero	Negative Zeros	Complex Zeros
i)	4	0	1	0
ii)	2	0	1	2
or iii)	0	0	1	4

Exercise 8-8-4

1. Investigate the nature of the zeros of each of the following expressions.

   a. $P(x) = x^9 + 3x^8 - 5x^3 + 4x + 6$

   b. $Q(x) = 2x^7 - 3x^4 + x^3 - 5$

   c. $R(x) = 3x^4 + 10x^2 + 5x - 4$

   d. $T(x) = 2x^3 + 5x^2 + x + 1$

   e. $Q(x) = x^4 - 2x^3 + 4x^2 - 3x + 1$

   f. $P(x) = x^{10} - 4x^6 + x^4 - 2x - 3$

   g. $T(x) = x^6 - x^2$

2. Show that the polynomial $R(x) = x^9 - x^5 + x^4 + x^2 + 1$ has at least six imaginary roots.

3. Show that the polynomial $P(x) = x^n - 1$ has exactly two real roots if n is even and only one real root if n is odd.

8-9 Polynomials with Integral Coefficients.

Reviewing what we know about factoring polynomials we now can get some idea about the number of real zeros. Furthermore, we can determine to some extent whether these zeros are zero, positive or negative. We also know that a polynomial with real coefficients and a zero of $a + bi$, $b \neq 0$ has $a + (-b)i$ as a zero. That is complex zeros appear in polynomials with real coefficients as conjugate pairs.

Through this point in the development of polynomials we cannot factor an n^{th} degree polynomial, $n > 2$, unless we are given $n - 2$ of the factors or $n - 2$ of the zeros. Without this information guessing zeros would be the only means of reaching complete factorization.

There exists a theorem about the rational zeros of a polynomial over the field of integers. Its application narrows down the "guesses" necessary if a given polynomial has rational zeros.

Theorem 8-9-1 Rational Zero Theorem

The polynomial $P(x)= a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$ where a_i, $i = o$ to n, is an integer, has a rational zero $\frac{r}{s}$ if and only if r is a factor of a_0 and s is a factor of a_n.

Example 8-9-2

Factor the polynomial $P(x) = x^4 + 2x^3 + x + 2$ as completely as possible.

Since $P(x)$ has no variation in sign there are no positive zeros.

$P(-x) = x^4 - 2x - x + 2$, hence, there are either two or no negative zeros.

Since the coefficients of $P(x)$ are integers if there are rational zeros they will be of the form r/s where r is a factor of 2 and s is a factor of 1. This means that r/s may be

$$\frac{2}{1}, -\frac{2}{1}, \frac{1}{1}, \text{ or } -\frac{1}{1}$$

Remembering that there are no positive zeros eliminates +2 and +1 as possibilities. Lets check -2 and -1.

$$\begin{array}{r|rrrrr} -2 & 1 & 2 & 0 & 1 & 2 \\ & & -2 & 0 & 0 & -2 \\ \hline & 1 & 0 & 0 & 1 & 0 \end{array}$$

$\therefore$ $(x - (-2))$ is a factor

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

$\therefore$ $(x - (-1))$ is a factor

This means $P(x) = (x + 2)(x + 1)(x^2 - x + 1)$

Now,
$$\begin{aligned} x^2 - x + 1 &= x^2 - x + \frac{1}{4} + 1 - \frac{1}{4} \\ &= (x - \frac{1}{2})^2 + \frac{3}{4} \\ &= (x - \frac{1}{2})^2 - (\frac{\sqrt{3}}{2}i)^2 \\ &= (x - \frac{1}{2} - \frac{\sqrt{3}}{2}i)(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i) \\ &= (x - (\frac{1}{2} + \frac{\sqrt{3}}{2}i))(x - (\frac{1}{2} - \frac{\sqrt{3}}{2}i)) \end{aligned}$$

$$\therefore \quad P(x) = (x + 2)(x + 1)(x - (\frac{1}{2} + \frac{\sqrt{3}}{2} i))(x - (\frac{1}{2} - \frac{\sqrt{3}}{2}i))$$

The previous example illustrates how Descartes' Rule of Signs may assist in narrowing down the possible trials which must be made after application of the Rational Zero Theorem.

At this time it should be pointed out that in some texts the zeros of a polynomial are referred to as roots. This means that in many books Theorem 8-9-1 is called the Rational Root Theorem. Hence, if r is a zero of a polynomial P(x) then r is a root of the equation P(x) = 0 and x - r is a factor of P(x).

Example 8-9-3

Factor $Q(x) = 3x^5 - 8x^4 - 8x^3 + 8x^2$ as completely as possible.

$$Q(x) = x^2 (3x^3 - 8x^2 - 8x + 8)$$

Note that 0 is a zero of multiplicity two.

Q(x) has integral coefficients which means that any rational zeros will be from the following set.

$$\{+\frac{8}{3}, -\frac{8}{3}, +\frac{4}{3}, -\frac{4}{3}, +\frac{2}{3}, -\frac{2}{3}, +\frac{1}{3}, -\frac{1}{3}, +8, -8, +4, -4, +2, -2, +1, -1\}$$

The student should determine what Descartes' Rule of Signs tells about the nature of the roots.

Trial and error shows that $\frac{2}{3}$ is a rational zero.

$$\therefore \quad Q(x) = x^2(x - \frac{2}{3})(x^2 - 2x - 4)$$

or $\quad Q(x) = x^2(x - \frac{2}{3})(x - (1 + \sqrt{5}i))(x -(1 - \sqrt{5}i)).$

Exercise 8-9-4

1. Factor the following polynomials as completely as possible.

   a. $P(x) = x^4 + 2x^3 - 12x^2 - 10x + 3$

   b. $T(x) = x^3 - 3x^2 - 12x + 54$

   c. $P(x) = x^6 + 4x^5 - 59x^4 + 4x^3 - 60x^2$

   d. $Q(x) = 6x^4 - 7x^3 + 8x^2 - 7x + 2$

2. Construct a flow chart and write a subsequent program which will check for all rational zeros of a polynomial given its coefficients and print the factored form.

3. Use the polynomials of exercise 1 to check this program in exercise 2.

4. Use the program to evaluate the rational zeros of the following polynomials.

   a. $P(x) = 6x^3 - 27x^2 + 37x - 100$

   b. $R(x) = 5x^4 + 8x^3 - 175x^2 - 288x - 180$

   c. $Q(x) = 16x^4 + 175x^2 - 193x - 12$

   d. $P(x) = 2x^4 + 13x^3 - 135x^2 - 54x - 216$

8-10 Rational Approximations of Irrational Zeros.

The previous section discusses complex zeros and real, rational zeros of a polynomial. Consider the polynomial

$$P(x) = x^3 + 2x - 5.$$

Descartes' Rule of Signs indicates there is exactly one positive zero. However, it is not rational, why?

There are several ways of getting an accurate approximation to an irrational root such as that of $P(x) = x^3 + 2x - 5$. These methods can involve laborious hand calculations if a high degree of accuracy is desired for the approximation. The computer, again, aids us in calculating such approximations. Two methods will be presented in this text while other approaches may be found in texts dealing with numerical analysis and computer programming.

We know from previous work in this chapter that for a polynomial P(x) if $P(a) < 0$ and $P(b) > 0$ $a,b \varepsilon R$ then there exists at least one real zero, c, between a and b. The most straightforward approach to approximating this real number c, is called the Bisection Method. This method is quite similar to the method used in Chapter 1 for finding an approximation for the square root of a number.

Consider the previous polynomial

$$P(x) = x^3 + 2x - 5$$

When we evaluate P(x) at 1, we find $P(1) = -2$. When we evaluate P(x) at 2, we obtain $P(2) = +7$. Our conclusion is that there exists at least one real zero, c, such that $1 < c < 2$. In this example, c is the only positive zero. Since $P(1) < 0$ and $P(2) > 0$, to calculate the first approximation to this zero we average the numbers 1 and 2.

$$c = \frac{a + b}{2} = \frac{1 + 2}{2} = 1.5$$

Evaluating P(1.5):

$$\begin{array}{r|rrrr} 1.5 & 1 & +\ 0 & +\ 2 & -\ 5 \\ & & 1.5 & 2.25 & 6.375 \\ \hline & 1 & 1.5 & 4.25 & 1.375 \end{array}$$

Now $P(1.5) = 1.375$ which means $P(1.5) > 0$. We now know that the zero lies between 1 and 1.5. Therefore, this interval is bisected and 1.25 is used as a second approximation. If $P(1.25) < 0$ then we know that the zero, c, is in the interval $1.25 < c < 1.5$. If $P(1.25) > 0$ then $1 < c < 1.25$. The process is then repeated until the desired approximation for the zero is reached.

Example 8-10-1

Construct a flow chart of the Interval Bisection algorithm for approximating irrational zeros of a polynomial.

The program for synthetic substitution that was written in Section 8-2 of this text will be used to find the values of a, b and c. The values desired for a and b are such that $P(a) > 0$ and $P(b) < 0$ or $P(a) < 0$ and $P(b) > 0$, and $c = \frac{a + b}{2}$.

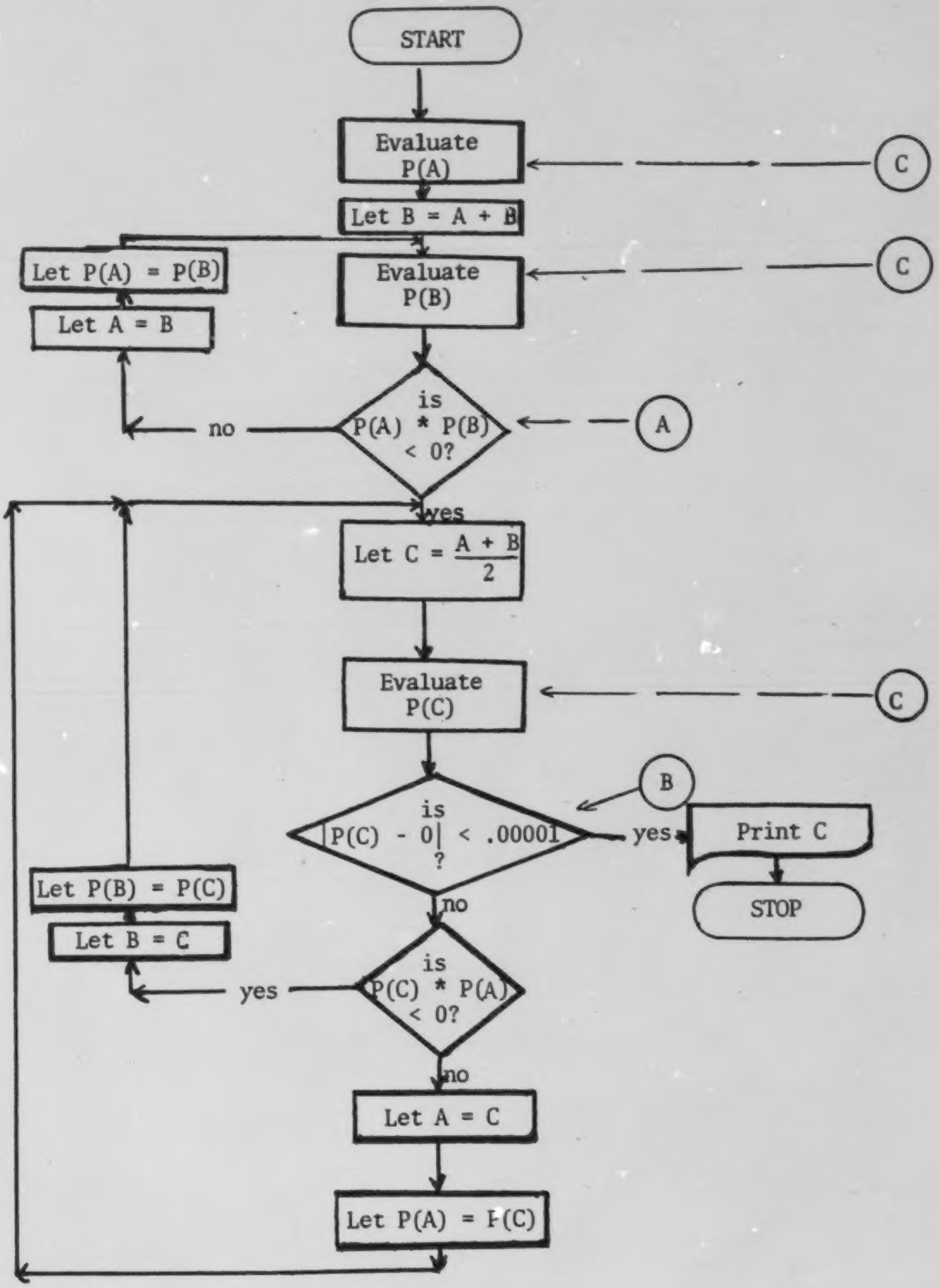

START
Evaluate P(A)
Let B = A + B
Let P(A) = P(B)
Let A = B
Evaluate P(B)
C
C
is P(A) * P(B) < 0?
no
yes
A
Let C = $\frac{A + B}{2}$
Evaluate P(C)
C
B
is |P(C) - 0| < .00001 ?
yes
Print C
STOP
no
Let P(B) = P(C)
Let B = C
is P(C) * P(A) < 0?
yes
no
Let A = C
Let P(A) = P(C)

Figure 8-10-2

There are three points of discussion regarding Figure 8-10-2.

Ⓐ By asking if P(A) * P(B) < 0 we are determining whether the values of P(A) and P(B) have the same sign. If P(A) < 0 and P(B) > 0 the product P(A) * P(B) < 0. If P(A) > 0 and P(B) < 0 the product P(A) * P(B) < 0. However, if P(A) < 0 and P(B) < 0 or if P(A) > 0 and P(B) > 0 then P(A) * P(B) > 0.

Ⓑ We are seeking a rational approximation for this irrational zero. Hence, P(C) will never actually become zero.

For the case in which the zero, C, in the interval a < c < b is rational we may obtain a value, P(C), which is zero, or it too may be an approximation. This rational approximation to a rational zero is caused by round-off error in averaging A and B. See Section 4-9, Exercise 4-9-10, Problem 3, Page 4-67.

The student may check to see whether or not the approximate zero is rational or irrational by using the Rational Zero Theorem.

Before going to the third point regarding this algorithm trace the flow chart as directed below.

Exercise 8-10-3

Trace the flow chart, Figure 8-10-2, using the a and b illustrated on the following graph of polynomial P(x), Figure 8-10-4. Place a mark on the graph each time a value for C is found and change the name for the points when directed by the algorithm. Complete five cycles before terminating this exercise.

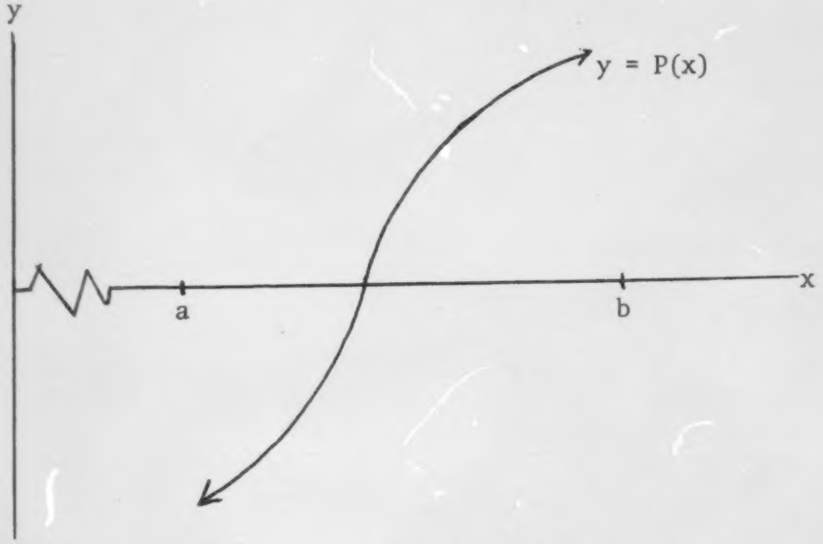


Figure 8-10-4

Ⓒ Notice the evaluation of the polynomial takes place for three different real numbers in our flow chart. A helpful programming procedure will be used in writing the program. This is the subroutine procedure and the related commands are:

GO SUB . . . RETURN

The purpose of these commands is to eliminate the need to repeat a frequently used portion of a program.

GO SUB transfers control to the specified statement number. The RETURN command transfers control to the statement immediately following the GO SUB statement which transferred control initially.

Example 8-10-5

Program:

```
  10 Let A = 10

  20 Print A

  30 GO SUB 100

  40 Print A

  50 GO SUB 100

  60 Print A

* 70 STOP

 100 Let A = A + 10

 110 RETURN

 120 END
```

Output:

```
10

20

30
```

*Line 70, STOP, is used as a step preceeding a subroutine to prevent accidental entry into the subroutine.

Example 8-10-6

Write a program which corresponds to the flow chart in Figure 8-10-2. This program should seek the positive irrational root of $P(x) = x^3 + 2x - 5$.

```
10  DIM A[20]
20  READ N
30  FOR I=N+1 TO 1 STEP -1
40  READ A[I]
50  NEXT I
60  LET A=0
70  LET X=A
80  GOSUB 340
90  LET R=P
100  IF ABS(P-0) <= .00001 THEN 320
110  LET B=A+1
120  LET X=B
130  GOSUB 340
140  IF ABS(P-0) <= .00001 THEN 320
150  LET S=P
160  IF R*S<0 THEN 200
170  LET A=B
180  LET R=S
190  GOTO 110
200  LET C=(A+B)/2
210  LET X=C
220  GOSUB 340
230  IF ABS(P-0) <= .00001 THEN 320
240  LET T=P
250  IF R*T<0 THEN 290
260  LET A=C
270  LET R=T
280  GOTO 200
290  LET B=C
300  LET S=T
310  GOTO 200
320  PRINT X,P
330  STOP
340  LET P=0
350  FOR I=N+1 TO 2 STEP -1
360  LET P=(P+A[I])*X
370  NEXT I
380  LET P=P+A[1]
390  RETURN
400  DATA 3,1,0,2,-5
410  END


RUN


 1.32827          8.58307E-06

DONE
```

In the program, lines 80, 130, and 220 access the subroutine found in lines 340-390, while lines 100, 140, and 230 check the functional value of the approximation to see if it is within .00001 of the desired functional value, zero.

The output first lists the rational approximation to the irrational zero of the polynomial $x^3 + 2x - 5$. The second figure 8.58307E-06 is the functional value of the approximation. This value is within the specified tolerance as 8.58307E-06 = .00000858307 which is within .00001 of zero.

Exercise Set 8-10-7

1. Using the program from Example 8-10-6, find an approximation for the position real zero in each of the following polynomial functions.

$$\{(x,y)|y = x^3 - 2x^2 + x - 3\}$$

$$\{(x,y)|y = 3x^4 - 4x^3 - 2x^2 - 1\}$$

2. Expand the program from Example 8-10-6 to approximate more than one real zero of a given polynomial function select arbitrary values for the lower and upper bounds of the interval in which the roots may be found.

3. Using the program from the previous exercise, find all approximations to the zeros of each of the following polynomial functions. From information gained by applying Descartes' Rule of Signs and the Rational Zero theorem to these functions, determine which approximations in the output of the program are approximations to rational zeros. Name these rational zeros.

$$\{(x,y)|y = 4x^3 + 13x + 6\}$$

$$\{(x,y)|y = x^2 - 2\}$$

$$\{(x,y)|y = x^4 - 2x^3 - 5x^2 + 10x - 3\}$$

$$\{(x,y)|y = 4x^6 - 66x^4 - 83x^3 + 216x^2 + 396x + 60\}$$

4. Using the program from exercise 2, find the rational approximation to the rational roots of $\{(x,y) | y = x^3 + 2x^2 - 4x - 8\}$. Graph this function. Explain why the program determines approximations to only one of the three real zeros.

5. In reference to exercise 4, above, what would have to be the nature of the roots of a general polynomial function

$$\{(x,y)\ y = a_nx + a_{n-1}x^{n-1} + \ . \ . \ . \ + a_1x + a_0\}$$

if the program were to successfully locate all real roots.

8-11 Polynomial Equations

Associated with every polynomial P(x) is the polynomial function defined by $y = P(x)$. Until now we have been concerned with finding the factors of polynomials as well as the zeros of polynomial functions. In order to find the zeros of a polynomial function $y = P(x)$, the solutions of the equation $P(x) = 0$ must be found.

Example 8-11-1

Find the zeros of the polynomial function

$$P(x) = x^3 + \frac{31}{6}x^2 + \frac{22}{3}x + \frac{5}{2}$$

To do this, the solution of the equation $P(x) = 0$ must be found. Therefore, let

$$x^3 + \frac{31}{6}x^2 + \frac{22}{3}x + \frac{5}{2} = 0$$

$$\therefore \ 6x^3 + 31x^2 + 44x + 15 = 0$$

Applying the Rational Zero Theorem we find that

$$6x^3 + 31x^2 + 44x + 15 = (x + \frac{1}{2})(x + \frac{5}{3})(x + 3)$$

Hence, $$(x + \frac{1}{2})(x + \frac{5}{3})(x + 3) = 0$$

and, $-\frac{1}{2}$, $-\frac{5}{3}$, -3 are zeros of P(x). Furthermore, $-\frac{1}{2}$, $-\frac{5}{3}$ and -3 are solutions or roots of the equation

$$x^3 + \frac{31}{6}x^2 + \frac{22}{3}x + \frac{5}{2} = 0.$$

There are many opportunities to solve for the roots of equations which involve polynomials. Some instances of polynomial equations are more complex than merely equating a polynomial with the zero polynomial. This fact is taken into account in the following definition of a polynomial equation.

Definition 8-11-2 Polynomial Equation

An equation $P(x) = Q(x)$ is a polynomial equation if and only if $P(x)$ and $Q(x)$ are polynomials.

Example 8-11-3

Find the solutions of the polynomial equation

$$x^3 + 3x^2 - 4 = x^3 + 2x^2 - 3$$

Utilizing the transformation principles for equations,

$$x^3 + 3x^2 - 4 - (x^3 + 2x^2 - 3) = 0$$

$$x^2 - 1 = 0.$$

By conventional technique we know that $x^2 - 1 = 0$ if and only if $x = 1$ or $x = -1$. Therefore, the solutions of the equation $x^3 + 3x^2 - 4 = x^3 + 2x^2 - 3$ are 1 and -1.

We know that the system of polynomials is closed under the operation of addition. This closure assures us that from any polynomial equation

$$P(x) = Q(x)$$

a new polynomial, $R(x)$, may be found by addition of $-Q(x)$ to both sides of the equation. Since

$$P(x) = Q(x)$$

then $$R(x) = 0 \quad \text{where} \quad R(x) = P(x) - Q(x)$$

The obvious conclusion to be drawn from this discussion is that the solution of the polynomial equation $P(x) = Q(x)$ is equivalent to the solution to $R(x) = 0$ where $R(x) = P(x) - Q(x)$. The set of solutions to $R(x) = 0$ is equivalent to the set of zeros of the polynomial $R(x)$.

Exercise 8-11-4

Solve the following polynomial equations.

1. $x^3 - 62x^2 + 122x - 101 = 19$
2. $x^3 - 7x + 7 = 0$
3. $4x^3 + 7x^2 + x + 4 = 3x^3 + 7x^2 + 13x - 4$
4. $3x^3 - 2x^2 - 10x - 3 = -(x^3 + 2x^2 - 8x)$
5. $x^3 + 5x - 5 = -2(3x^3 - x - 1)$
6. $x^4 + 100x^2 - 200x + 129 = 20x^3 - x^2 + 92x - 1$
7. $x^4 - 26x^2 + 48x - 52 = x^2 - 4$

If the computer was used to find the solutions in the previous exercise, some calculation was probably done by hand before entering data in the program. At this time two objectives will be outlined. One is a computer algorithm for solving equations which will not necessitate any hand calculation prior to running the program. The second will involve a method of finding solutions which is different from the Interval Bisection Technique. This new technique will be called the Marching Technique.

Initially, consider the polynomial equation

$$P(x) = Q(x)$$

There are two functions suggested by this equation:

$$(1) \qquad y = P(x)$$

and

$$(2) \qquad y = Q(x)$$

The solutions to the equation $P(x) = Q(x)$ are those replacements of x for which the functional values of (1) on the same coordinate plane, the solutions can be easily seen as the x - coordinates of the points of intersection. See Figure 8-11-5.

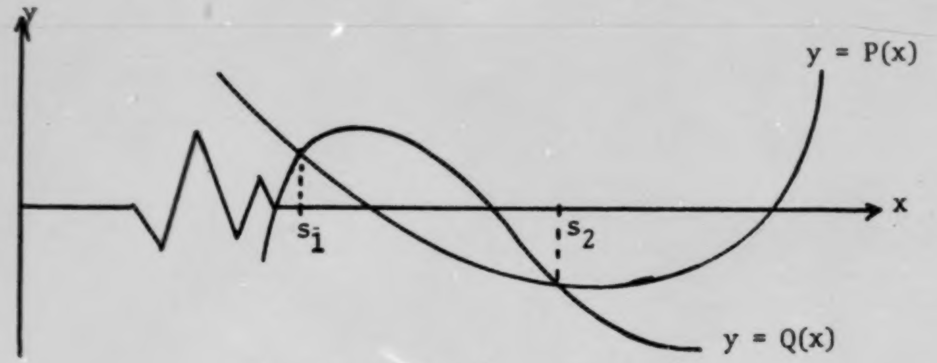


Figure 8-11-5

Figure 8-11-5 shows the solutions of the equation $P(x) = Q(x)$ are S_1 and S_2.

To find the value of S_1 and S_2 look at Figure 8-11-6 which is an enlargement of the region in which S_1 is located.

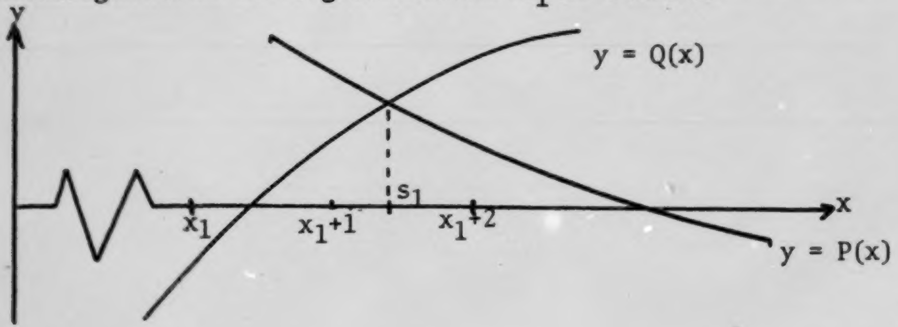


Figure 8-11-6

The Marching Technique for evaluating the approximation to S_1 begins much like the Interval Bisection Technique used to approximate zeros of functions in the preceeding section. First, select an arbitrary replacement for the variable which is less than the desired solution. Figure 8-11-6 illustrates this initial replacement x_1. Using this value, x_1, trace through the following flow chart of the algorithm. Sketch the results from this activity on the graph provided in Figure 8-11-6.

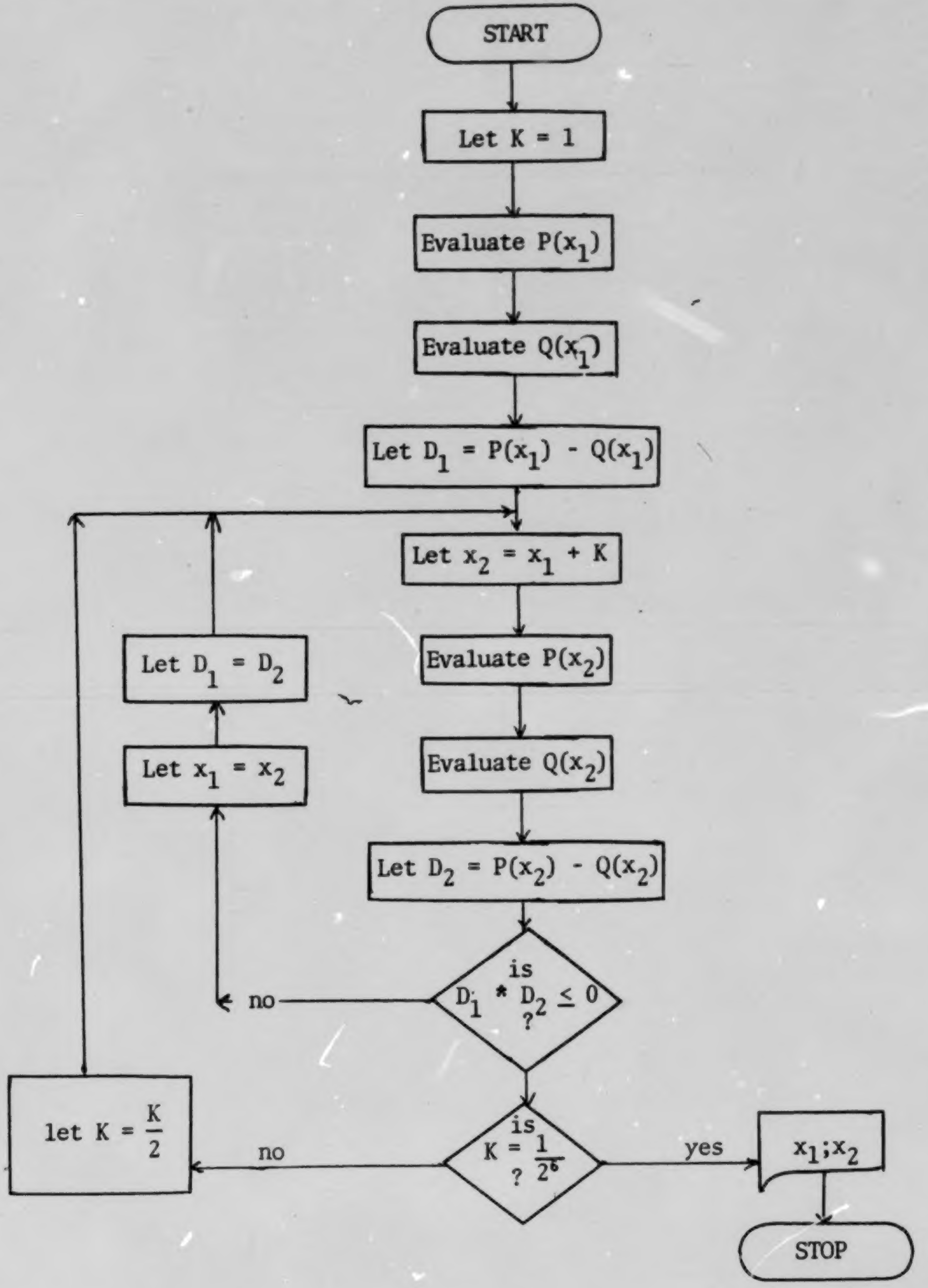

START
Let K = 1
Evaluate $P(x_1)$
Evaluate $Q(x_1)$
Let $D_1 = P(x_1) - Q(x_1)$
Let $x_2 = x_1 + K$
Evaluate $P(x_2)$
Evaluate $Q(x_2)$
Let $D_2 = P(x_2) - Q(x_2)$
is $D_1 * D_2 \leq 0$?
no
Let $x_1 = x_2$
Let $D_1 = D_2$
is $K = \frac{1}{2^6}$?
no
let $K = \frac{K}{2}$
yes
$x_1;x_2$
STOP

Figure 8-11-7

The flow chart illustrates the Marching Technique. It may be used in place of the Interval Bisection Technique for root search. However, in this case it is used to find approximate values for x which yield values for P(x) and Q(x) which are quite close to one another. The actual value for x is within the interval $x_1 \leq x \leq x_2$. In this program a tolerance on the functional values is not used. Instead, we print values of x_1 and x_2 when a sign change occurs in the difference P(x) - Q(x). At such a time x_1 and x_2 are printed after K has been made as small as $\frac{1}{2^6}$. This power of two is used to minimize machine error in rounding off.

Exercise 8-11-8

1. Write the computer program which corresponds to the flow chart in Figure 8-11-7. Expand the program to recycle K to the value of 1 and search for additional solutions after the first one is computed.
2. Use this new program to find approximate solutions to the equation in Exercise 8-11-4. Check these solutions with the results obtained in Exercise 8-11-4 by the program used in Section 8-11.
3. Can you illustrate graphically two polynomial functions for which there exists an x such that P(x) = Q(x) which the Marching Technique will fail to find? How would the graph of R(x) behave if R(x) = P(x) - Q(x)?

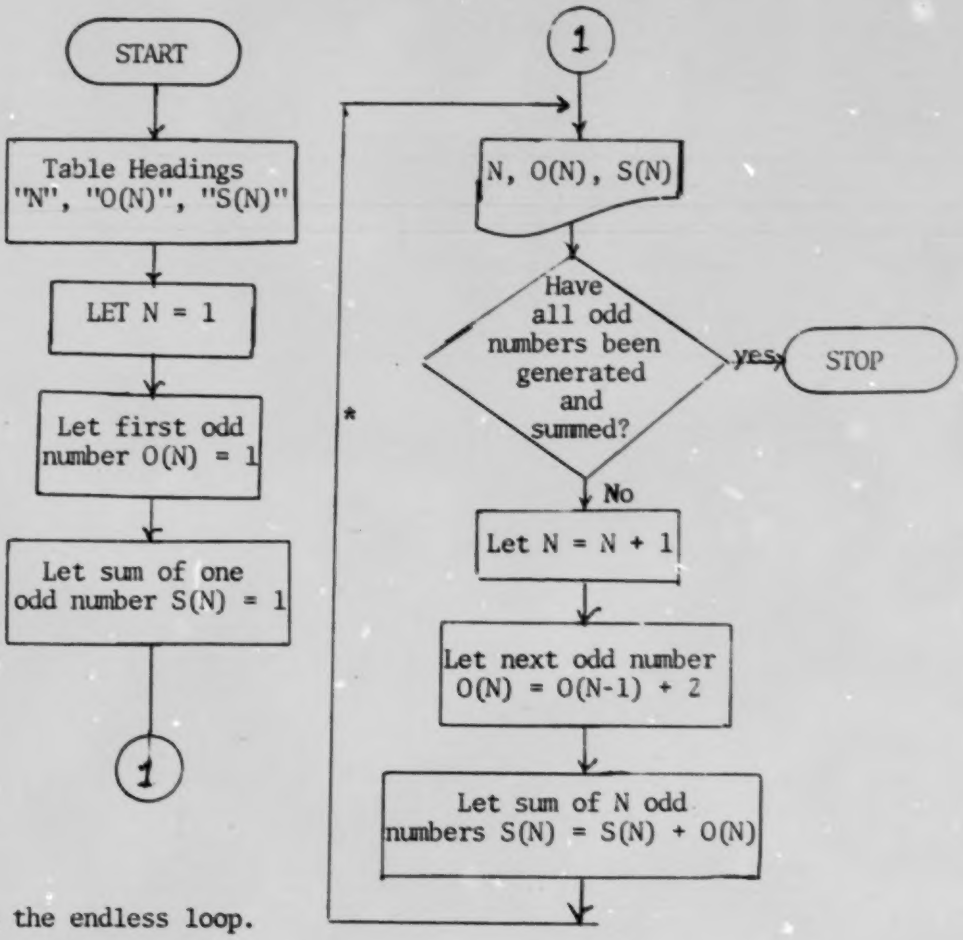
Chapter 9

Sequences

9-1 Introduction

Suppose that we have at our disposal a computer which has the capacity to manipulate and print natural numbers of any magnitude and to run indefinitely without interruption. We will ask this unusual computer to print a table of the set of odd numbers and the sum of all odd numbers. To perform this assignment we need a program that will generate the odd numbers successively beginning with 1, keep a running total as each new odd number is generated and keep count of the number of odd numbers generated and summed.

The following flow chart illustrates the logic of one approach to our task. We will use N to represent the number of odd numbers, O(N) will represent n^{th} odd number and S(N) the sum of n odd numbers.


START
Table Headings "N", "O(N)", "S(N)"
LET N = 1
Let first odd number O(N) = 1
Let sum of one odd number S(N) = 1
1
1
N, O(N), S(N)
Have all odd numbers been generated and summed?
yes
STOP
No
Let N = N + 1
Let next odd number O(N) = O(N-1) + 2
Let sum of N odd numbers S(N) = S(N) + O(N)
*

*Note the endless loop.

Figure 9-1-1

```
                    OUTPUT

n                   O(n)                   S(n)

1                   1                      1
2                   3                      4
3                   5                      9
4                   7                      16
5                   9                      25
.                   .                      .
.                   .                      .
.                   .                      .
K                O(K)=O(K-1)+2       S(K)=S(K-1)+O(K)
.                   .                      .
.                   .                      .
.                   .                      .
```

Figure 9-1-2

The above output table defines two functions:

$f = \{(1,1),(2,3),(3,5),(4,7),\ldots(K,\ O(K-1)+2),\ldots\}$ and

$g = \{(1,1),(2,4),(3,9),(4,16),\ldots(K, S(K-1)+O(K)),\ldots\}$

Functions of the type f and g are called sequences. In general a sequence is a function which places the set of natural numbers in one-to-one correspondence with any set of second elements. In this book we will consider only those sequences whose ranges are subsets of the set of real numbers.

Definition 9-1-3 Sequences

A sequence is a function $f = \{(1,A_1),(2,A_2),(3,A_3),\ldots(n,A_n)\ldots\}$ where $A_n \varepsilon R$ and the domain is the set of natural numbers.

For convenience $\{(1,A_1),(2,A_2),(3,A_3),\ldots(n,A_n),\ldots\}$, will be abbreviated to $\{(n,A_n)\}$.

Example 9-1-4

$\{(1, \frac{1}{1}), (2, \frac{1}{2}), (3, \frac{1}{3}), \ldots (n, \frac{1}{n})\ldots\}$ will be abbreviated to $\{(n, \frac{1}{n})\}$.

The second elements of a sequence are called TERMS. When we speak of the first term of a sequence we mean the value of the function for the argument 1. In general, for each n, the n^{th} term if a sequence is the value of the function for the argument n.

Example 9-1-5

The terms of the sequence $\{(n, 3n-5)\}$ are $\{-2,1,4,7,10,\ldots\}$.

Consider the functions:

(a) $\{(n,2n)\} = \{(1,2)(2,4)(3,6)\ldots\}$ and

(b) $\{(n,R_n) | R_n \text{ a random number}\} = \{(1,-1)(2,\frac{1}{2})(3,\pi)(4,.893)\ldots\}$

Function (a) and (b) above are both sequences. In (a) a particular real number is associated with each natural number while in (b) each second element is not necessarily unique to a particular first element.

Exercise 9-1-6

The computer has the capability to generate random numbers R_n such that $0 < R_n < 1$.

Refer to your BASIC manual for the proper command.

Write a program that will print:

(a) The first 10 elements of the sequence $\{(n,R_n)\}$

(b) The first 10 elements of the sequence $\{(n,Y_n) | 1 < Y_n < 10\}$ using the random number function.

Note that Definition 9-1-3 defines an infinite set of ordered pairs. Much of our work will deal with only a finite number of elements of a sequence.

Example 9-1-7

Write the first five elements of $\{(n, \frac{1}{2n})\}$.

Solution: $\{(1, \frac{1}{2}), (2, \frac{1}{4}), (3, \frac{1}{6}), (4, \frac{1}{8}), (5, \frac{1}{10})\}$

Exercise 9-1-8

Write out the first 5 elements of each of the following sequences.

1. $\{(n, 3n - 5)\}$
2. $\{(n, 2^n)\}$
3. $\{(n, \frac{n-1}{n})\}$
4. $\{(n, 2n - 1)\}$
5. $\{(n, n^2)\}$

Refer again to the output tables in Figure 9-1-2.

n	O(n)	S(n)
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
.	.	.
.	.	.
.	.	.
K	O(K) = ?	S(K) = ?
.	.	.
.	.	.
.	.	.

Figure 9-1-9

Problem 4 and 5 of Exercise 9-1-8 should provide the hint that O(K) and S(K) could now be defined explicity as $O(K) = 2K - 1$ and $S(K) = K^2$. Therefore, $2k - 1$ and k^2 represent the k^{th} (or general) term of the sequences:

$$\{(K,O(K))\} = \{(1,1)(2,3)(3,5)\ldots(k,2k - 1)\ \ldots\} = \{(K,\ 2K - 1)\}$$

and

$$\{(K,S(K))\} = \{(1,1)(2,4)(3,9)\ldots(k,k^2)\ldots\} = \{(K,K^2)\}$$

Exercise 9-1-10

1. Flow chart a program to print K, O(K) and S(K) using the above explicit definitions of O(K) and S(K), $1 \leq K \leq N$.

2. Which program is the more efficient, yours or the one flow charted in Figure 9-1-1? Why?

3. Below you will find the first several terms of various sequences:

   (a) Predict a general term for each.

   (b) Write the ordered pair notation of the sequence from which each set of terms comes.

      (1) 2, 4, 6, 8, ...

      Solution: (a) 2n

      (b) $\{(n, 2n)\}$

      (2) -3, -6, -9, -12, ...

      (3) 3, 5, 7, 9, ...

(4) $2, 5, 8, 11, \ldots$

(5) $-1, -4, -7, -10, \ldots$

(6) $\frac{3}{4}, \frac{6}{7}, \frac{9}{10}, \frac{12}{13}, \ldots$

(7) $2, 4, 8, 16, \ldots$

(8) $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$

(9) $-2, 1, 4, 7, \ldots$

(10) $\frac{1}{4}, 1, \frac{9}{4}, 4, \ldots$

9-2 Partial Sums

The program to print all odd numbers generated a sequence of sums by finding successively the sums for one, two, three, ... K terms of the sequence as follows:

Number of Terms	Indicated Sum of N Terms	Sum
1	1	1
2	1 + 3	4
3	1 + 3 + 5	9
4	1 + 3 + 5 + 7	16
5	1 + 3 + 5 + 7 + 9	25
.	.	.
.	.	.
.	.	.
K	1 + 3 + 5 + 7 + 9 + ... + 2K-1	K^2
.	.	.
.	.	.
.	.	.

Figure 9-2-1

The sequence $\{(n, n^2)\} = \{(1,1)(2,4)(3,9)(4,16) \ldots (k,k^2) \ldots\}$ is a sequence of partial sums of the sequence $\{(n, 2n - 1)\}$ because each term of the sequence $\{(n, n^2)\}$ is the sum of a finite number of the terms of $\{(n, 2n - 1)\}$.

We will have need to deal with partial sums of sequences, therefore, we now adopt the following convenient notation.

The Greek letter $\sum$ (sigma), called the summation sign, is used to abbreviate an indicated sum. For instance, from Figure 9-1-2, we can write

$$1 + 3 + 5 + 7 + 9$$

in the form (2(1)-1) + (2(2)-1) + (2(3)-1) + (2(4)-1) + (2(5)-1).

It is evident that the general term is 2k-1, so we designate the indicated sum 1 + 3 + 5 + 7 + 9

by $\displaystyle\sum_{k=1}^{5} 2k-1$

read "the sum of 2k - 1 from k = 1 to k = 5" or "summation 2k - 1 from 1 to 5."

Example 9-2-2

(a) Express $\displaystyle\sum_{k=1}^{6} \frac{1}{2}k$ in expanded form.

Solution:

$$\sum_{k=1}^{6} \frac{1}{2}k = \frac{1}{2}(1) + \frac{1}{2}(2) + \frac{1}{2}(3) + \frac{1}{2}(4) + \frac{1}{2}(5) + \frac{1}{2}(6)$$

$$= \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3$$

(b) Write $\frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5}$ in sigma notation.

Solution: $\displaystyle\frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} = \sum_{k=1}^{5} \frac{2}{k}$

Using the sigma notation the conclusion that the sum of n odd numbers is n^2 can be expressed as:

$$\sum_{k=1}^{n} (2k-1) = n^2$$

Exercise 9-2-3

1. Expand each of the following:

(a) $$\sum_{m=1}^{5} m$$

(b) $$\sum_{q=1}^{b} (-1)^q$$

(c) $$\sum_{p=1}^{4} \frac{p(p-1)}{2}$$

(d) $$\sum_{m=1}^{5} \frac{1}{m(m+1)}$$

(e) $$\sum_{q=1}^{5} \left(\frac{1}{q} - \frac{1}{q+1}\right)$$

(f) $$\sum_{n=1}^{5} \frac{1}{n^2}$$

2. Write each of the following using $\sum$ notation:

(a) $-1 + 1 + 3 + 5 + \ldots + 17$

(b) $2 + (-4) + 8 + (-16) + \ldots + (-256)$

(c) $(1.3) + (2.4) + (3.5) + \ldots$

(d) $1 + 8 + 27 + 64 + \ldots$

(e) $(1 - 3 \cdot 1^2) + (1 - 3 \cdot 2^2) + (1 - 3 \cdot 3^2) + \ldots + (1 - 3 \cdot 6^2)$

(f) $(-3) + (-6) + (-9) + (-12) + \ldots$

3. (a) Expand:

$$\sum_{k=1}^{9} \frac{k(k+1)}{2}$$

(b) Write $1 + 5 - 9 - 13 + 17 + 21 - 25 - 29 + \ldots$ in notation. The result of part (a) above should be of some assistance to you.

4. Write each of the following in sigma notation.

   a. For each $n \in N$, the sum of the numbers $2k + 1$, for all $k \leq n$, $k \in N$, is $n^2 + 2n$.

   b. For each $n \in N$, the sum of the numbers $3p - 2$, for all $p \leq n$, $n \in N$, is:

   $$\frac{n(3n-1)}{2}.$$

   c. The sum of k natural numbers, $k = 7$, is 28.

If we know an expression for the general (k^{th}) term of a sequence, it is possible to write a more efficient program to find the partial sums of a sequence.

Example 9-2-4

The sequence of partial sums of the sequence $\{(n, \frac{1}{2^n})\}$ is:

$n = 1$: $\quad \frac{1}{2} \quad = \frac{1}{2}$

$n = 2$: $\quad \frac{1}{2} + \frac{1}{4} \quad = \frac{3}{4}$

$n = 3$ $\quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \quad = \frac{7}{8}$

$n = 4$ $\quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

$n = k$: $\quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \frac{1}{2^k} = ?$

Figure 9-2-5

Programming the computer to form K terms of the sequence $\{(n, \frac{1}{2^n})\}$ recursively and keeping a running sum of the terms would result in a program similar to the one below. We use C as the counting variable, T to represent terms of the sequence and S to represent the partial sums of the sequence of terms.

```
 10  INPUT K
 20  LET C = 0
 30  LET S = 0
 40  LET N = 1
 50  LET T = 1/2↑N
 60  LET S = S + T
 70  LET C = C + 1
 80  IF C = K THEN 110
 90  LET N = N + 1
100  GO TO  50
110  PRINT  "THE SUM OF"; K "TERMS IS"; S
120  END
```

Figure 9-2-6

By examining the last column of Figure 9-2-5 we can arrive at a general term for the sum of terms of the sequence $\{(n, \frac{1}{2^n})\}$. Do you see that the sum of:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k} \text{ is } \frac{2^k - 1}{2^k} ?$$

Now our program in Figure 9-2-6 can be reduced to:

```
10  INPUT K
20  PRINT "THE SUM OF"; K; "TERMS IS"; (2↑K - 1)/2↑K
30  END
```

Figure 9-2-7

Exercise 9-2-8

1. Convince yourself that the output of the programs of Figures 9-2-6 and 9-2-7 are identical by tracing through the operations performed by the computer.

2. Given the sequence $f = \{(n, \frac{1}{n(n+1)})\}$, using the approach used in Example 9-2-4:

   (a) Write a program to find the sum of k terms of f by using the recursive approach to generating the partial sums. (See Figure 9-2-6).

   (b) Determine a general term of the sequence of partial sums. Write the program to generate the partial sum explicitly. (See Figure 9-2-7).

9-3 Converging and Diverging Sequences

Let us refer again to the sequence $\mathbf{f} = \{(n, \frac{1}{2^n})\}$ and the sequence of its partial sums $g = \{(n, \frac{2^n - 1}{2^n})\}$.

a) $f = \{(1, \frac{1}{2}), (2, \frac{1}{4}), (3, \frac{1}{8}), (4, \frac{1}{16}), \ldots (k, \frac{1}{2^n}), \ldots\}$

b) $g = \{(1, \frac{1}{2}), (2, \frac{3}{4}), (3, \frac{7}{8}), (4, \frac{15}{16}), \ldots (k, \frac{2^k - 1}{2^k}), \ldots\}$

The graphs of (f) and (g) follow:

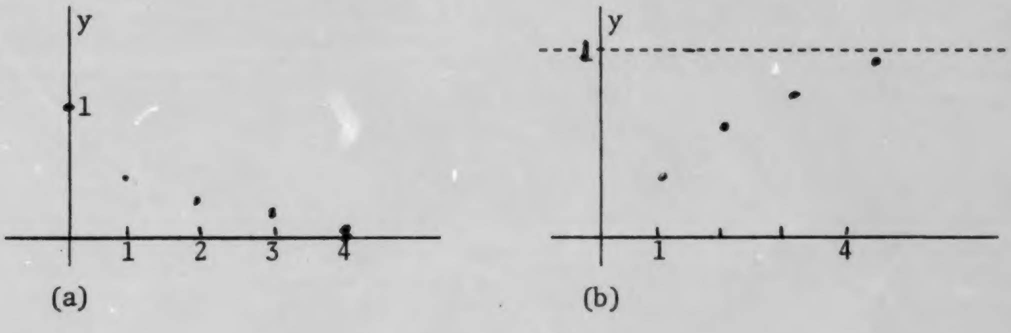


Figure 9-3-1

We note in (a) that as n becomes large the values of the function f, approach (come closer and closer to) 0. In (b) as n becomes large the value of the function g approaches 1. Sequences whose values approach some fixed number L as n becomes large are said to be convergent and the number L is said to be the limit of the sequence. Sequences that are not convergent are said to be divergent.

Example 9-3-2

$\{(n, 2^n)\} = \{(1,2)(2,4)(3,8)(4,16), \ldots (k, 2^k) \ldots\}$ is divergent because as n becomes large the terms of the sequence 2^n increase without bound.

The sequence $\{(n, (-2)^n)\} = \{(1,-2)(2,4)(3,-8)(4,16), \ldots (k, (-2)^k) \ldots\}$ is divergent because it has terms whose values oscillate from negative to positive and do not approach a fixed number.

Exercise 9-3-3

Below are the first several terms of different sequences. Tell which are terms of convergent sequences. If convergent, indicate the limit.

1. $0, 1, 0, 2, 0, 3, 0, 4, \ldots$
2. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$
3. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
4. $-1, 2, -3, 4, -5, \ldots$
5. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
6. $(1 + \frac{1}{2}), (1 + \frac{1}{2} + \frac{1}{4}), (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}), \ldots$
7. $1, \frac{1}{2}, -1, \frac{1}{3}, 1, \frac{1}{4}, -1, \frac{1}{5}, \ldots$
8. $0.6, 0.66, 0.666, 0.6666, \ldots$
9. $2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5}, \ldots$
10. $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots$

Example 9-3-4

Find the limit of $f = \{(n, \frac{3n+2}{n})\}$

Since $\frac{3n+2}{n} = 3 + \frac{2}{n} = 3 + 2(\frac{1}{n})$, the limit of f is equal to limit of $\{n, 3 + 2(\frac{1}{n})\}$. It seems logical to say that since $\frac{1}{n}$ approaches 0, as n becomes large, $2(\frac{1}{n})$ also approaches 0. Therefore, $3 + 2(\frac{1}{n})$ would approach 3 as n becomes large. We conclude then that the limit of f is 3.

Exercise 9-3-5

For each of the following sequences determine (a) whether the sequence is convergent or divergent (b) the limit of the convergent sequences.

1. $\{(n, \frac{8}{3n})\}$
2. $\{(n, \frac{6n - 4}{6n})\}$
3. $\{(n, \frac{3n + 5}{6})\}$
4. $\{(n, \frac{8 - 10n}{2n})\}$
5. $\{(n, 3(\frac{1}{2})^n)\}$
6. $\{(n, \frac{-2n - 9}{3n})\}$
7. $\{(n, (-\frac{3}{4})^n)\}$
8. $\{(n, -n^2 + 6n - 8)\}$
9. $\{(n, \frac{4 + 5n}{n})\}$
10. $\{(n, \frac{n - 5n}{n^3})\}$

9-4 Arithmetic Sequences

The sequence $f = \{(n, 2n - 1)\}$ whose terms are 1, 3, 5, 7, 9, ... is an arithmetic sequence. In arithmetic sequences the difference between any two successive terms is constant. This difference, called the common difference, is 2 for the above sequence. Any term, A_k of an arithmetic sequence can be generated by adding the common difference, d, to the A_{k-1} term. The terms of an arithmetic sequence are of the form (A_1), $(A_1 + d)$, $(A_1 + 2d)$, $(A_1 + 3d)$, ...

Definition 9-4-1 Arithmetic Sequence

A sequence $\{(n, A_n)\}$ is an arithmetic sequence if and only if for each $k \epsilon N > 1$, $A_k = A_{k-1} + d$, $d \epsilon R$. Arithmetic sequences are sometimes referred to as arithmetic progressions.

Example 9-4-2

$\{(1,3)(2,6)(3,9)(4,12) \ldots (k, 3k) \ldots\}$ is an arithmetic sequence. The common difference, $d = 3$.

Exercise 9-4-3

Fill in the blanks to get the terms of an arithmetic sequence. Indicate the common difference for each.

1. 2, 3½, 5, ___, ___, ___, ...
2. -2, 1, ___, 7, ___, ___, ...
3. 4, ___, ___, 16, ___, ___, ...
4. -8, ___, ___, ___, ___, ___, 17, ...
5. ___, ___, ___, 16, ___, ___, 32, ...
6. ___, ___, ___, ___, -30, ___, -4, ...
7. 6, ___, ___, ___, ___, ___, 7, ...
8. $\frac{1}{3}$, $\frac{5}{6}$, ___, ___, ___, ___, ...
9. -3, ___, $-3 + 4\sqrt{2}$, ___, ___, ...
10. ___, ___, ___, ___, 7, ___, ___, ...

11. Complete the following table:

n = 1	2	3	4	...	k	...
An = A_1,	$A_1 + d$,	$A_1 + 2d$,	$A_1 + ?$,	...	?	...

The k^{th} term of an arithmetic sequence with first term A_1 and common difference any real number d, is $A_k = A_1 + (k-1)d$.

Exercise 9-4-4

Each of the following are the first several terms of a sequence. Find the indicated term for each.

1. 7, 11, 15, ... , 70th term
2. -2, 1, 4, ... , 15th term
3. $\frac{5}{2}$, $\frac{13}{2}$, ... , 23rd term
4. (-2x, 2c - 3x, 4c - 4x, ... , 17th term

5. Write the first 5 terms of an arithmetic sequence in which the second term is m and the third term is p.

6. If the third term of an arithmetic sequence is -1 and the 16th is 11/2, what is the first term?

7. How many integers are there between 35 and 350 which are divisible by 23.

8. The arithmetic mean between two numbers a and b is $\frac{a + b}{2}$. Find the arithmetic mean between each of the following pairs of numbers..

   (a) $a = 5,\ b = 65$

   (b) $a = -6,\ b = 2$

   (c) $a = 3 - \sqrt{3},\ b = 7 + 5\sqrt{3}$

   (d) $a = (c + d)^2,\ b = c^2 - d^2$

9. Take every 5^{th} term from an arithmetic sequence and form a new sequence. Is the new sequence arithmetic?

10. If $3\frac{1}{2}$ and $8\frac{1}{2}$ are the first and eighth terms of an arithmetic sequence, find the six terms that should appear between these two so that all eight terms will be in arithmetic progression. (The six terms you are asked to find are called the six arithmetic means between $3\frac{1}{2}$ and $8\frac{1}{2}$).

Consider now the partial sums of an arithmetic sequence

$$\sum_{k=1}^{n} A + (K - 1)d = A_1 \qquad \text{when } n = 1$$

$$= A_1 + (A_1 + d) \qquad n = 2$$

$$= A_1 + (A_1 + d) + (A_1 + 2d) \qquad n = 3$$

$$= A_1 + (A_1 + d) + (A_1 + 2d) + (A_1 + 3d) \qquad n = 4$$

$$\vdots$$

$$= A_1 + (A_1 + d) + (A_1 + 2d) + \ldots + (A_1 + (n-1)d) \qquad n \in N$$

If we use S_k to represent the partial sum of k terms of an arithmetic sequence we have:

(1) $S_k = A_1 + (A_1 + d) + (A_1 + 2d) + \ldots + (A_1 + (k-2)d) + (A_1 + (k-1)d)$

Reversing the terms on the right side of the above equation we get:

(2) $S_k = (A_1 + (k-1)d) + (A_1 + (k-2)d) + \ldots + (A_1 + 2d) + (A_1 + d) + A_1$

Adding (1) and (2) we get:

(3) $2S_k = (2A_1 + (k-1)d) + (2A_1 + (k-1)d) + \ldots + (2A_1 + (k-1)d)$

(4) $2S_k = k\left[2A_1 + (k-1)d\right]$ Why?

(5) $S_k = \frac{k}{2}\left[2A_1 + (k-1)d\right]$

Formula (5) can be expressed in a different form:

(5) $S_k = \frac{k}{2}\left[2A_1 + (k-1)d\right]$

(6) $S_k = \frac{k}{2}\left[A_1 + A_1 + (k-1)d\right]$

But $A_k = A_1 + (k-1)d$

(7) $\therefore S_k = \frac{k}{2}(A_1 + A_k)$

The formulas (5) and (7) make it possible for us to do problems similar to the following examples.

Example 9-4-5

Find the sum of the first 27 odd numbers.

$A_1 = 1,\ d = 2,\ k = 27$

(5) $S_k = \frac{k}{2}\left[2A_1 + (k-1)d\right]$

$S_{27} = \frac{27}{2}\left[2(1) + (27-1)2\right]$

$$S_{27} = \frac{27}{2}(2 + 52)$$

$$S_{27} = 27(27)$$

$$S_{27} = 729$$

Find the sum of the first 32 terms of the arithmetic sequence whose first term is -5 and whose 32nd term is 27.

(7) $S_k = \frac{k}{2}(A_1 + A_k)$

$k = 32, A_1 = -5, A_k = 27$

$$S_{32} = \frac{32}{2}(-5 + 27)$$

$$S_{32} = 16(22)$$

$$S_{32} = 352$$

Exercise 9-4-6

1. Find the sum of the following:

   (a) $1 + 2 + 3 + \ldots + 10$

   (b) $1 + 2 + 3 + \ldots + 999$

   (c) $-3 - 6 - 9 - 12 - 15$

2. On a ship, time is marked by striking one bell at 12:30, two bells at 1:00, three bells at 1:30, etc. up to a maximum of 8 bells. The sequence of bells then begins anew, and it is repeated in each successive interval of four hours throughout the day. How many bells are struck during a day (24 hours)? How many are struck at 10:30 p.m.?

3. Find

$$\sum_{k=1}^{n}(ak + b)$$

4. Find n if

   $1 + 2 + 3 + \ldots + n = 153$

5. Find a and b if: $\sum_{k=0}^{4} (ak + b) = 10$ and $\sum_{k=1}^{4} (ak + b) = 14$

6. The digits of a three digit positive integer are in arithmetic progression and their sum is 21. If the digits are reversed, the new number is 396 more than the original number. Find the original number.

7. Find x if $(3 - x)$, $-x$, $9 - 2x$ are in arithmetic progression.

8. The sum of three numbers which are in arithmetic progression is -3 and their product is 8. Find the numbers.

9. Find the sum of all positive integers less than 300 which

   (a) are multiples of 7

   (b) end in 7.

10. (a) A grocer stacks canned goods in a triangular display like the one shown in the figure below. How many cans are there in a 10-high stack?

    (b) If the cans are shipped 24 to a case, what is the smallest number of cases he needs to make a triangular stack, assuming that he wishes to use all the cans in the cases? If such a stack is possible, how many rows does it have?

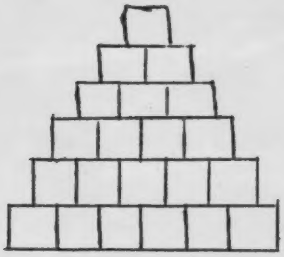


Figure 9-4-7

9-5 Geometric Sequences

Definition 9-5-1

A sequence $\{(n, A_n)\} = \{(1, A_1), (2, A_2), (3, A_3), (4, A_4), \ldots\}$ is a geometric sequence if and only if for each $k > 1 \epsilon N$, $A_k = A_{k-1} \cdot r$, $r \epsilon R$.

Example 9-5-2

$(n, (\frac{1}{2})^n) = \{(1, \frac{1}{2}) \; (2, \frac{1}{4}) \; (3, \frac{1}{8}) \; \ldots \; (k, A_{k-1} \cdot \frac{1}{2}) \; \ldots\}$ is a geometric sequence in which r is $\frac{1}{2}$. The number r is called the common ratio.

Exercise 9-5-3

Each of the following is a list of the first several terms of a geometric sequence. Determine the common ratio of each.

1. $1, 2, 4, 8, \ldots$
2. $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, 3\sqrt{6}$
3. $\frac{10}{3}, 1, .3, .09, 0.27, \ldots$
4. $\pi, -\pi^2, \pi^3, -\pi^4,$

In general if A_1 is the first term of a sequence and r is the common ratio, the terms of the sequence can be listed as shown below:

$n = 1 \;, 2 \;, 3 \;, 4 \;, 5 \;, \ldots \;, k \;, \ldots$

$A_n = A_1, \; A_1r^1, \; A_1r^2, \; A_1r^3, \; A_1r^4, \; \ldots, \; A_1r^{k-1} \ldots$

From the above table we observe the formula for finding the k^{th} term of a geometric sequence given the first term and the common ratio is

$$A_k = A_1 \cdot r^{k-1}.$$

Example 9-5-4

What is the 18th term of the sequence whose first terms are:

$-\frac{9}{8}, \frac{3}{4}, -\frac{1}{2}, \frac{1}{3}, \ldots$

Solution:

By dividing any one term by the term preceeding it we get $r = -\frac{2}{3}$.

Therefore, $A_{18} = A_1 \cdot r^{n-1}$

$$A_{18} = -\frac{9}{8}\left(-\frac{2}{3}\right)^{17}$$

$$= -\frac{9}{8}\left(-\frac{131,072}{129,140,163}\right)$$

$$= \frac{16,384}{14,348,907}$$

Exercise 9-5-5

1. Each of the following is a list of the first several terms of a sequence. For each, determine if the sequence is geometric. If so, find the term indicated. You may choose to use the computer to carry out the assignment.

   (a) $-5, -10, -20, \ldots$, 18th term

   (b) $6, \frac{12}{5}, \frac{24}{25}, \ldots$, 7th term

   (c) $\frac{1}{4}, 1, 4, \ldots$, 9th term

   (d) $\frac{1}{2}, \frac{3}{8}, \frac{9}{32}, \ldots$, 6th term

   (e) $-\frac{5}{2}, \frac{1}{2}, -\frac{1}{10}$, 8th term

   (f) $1, \frac{3}{4}, \frac{1}{2}, \ldots$, 16th term

   (g) $7, \frac{7}{8}, \frac{7}{64}, \ldots$, 15th term

2. How many terms in the geometric sequence whose terms are 32, 16, 8, ..., and whose last term is $\frac{1}{256}$?

Now let us see if we can determine a formula for finding the partial sums of a geometric sequence.

$$\sum_{k=1}^{n} A_1 r^{k-1} = A_1 \qquad \text{When } n = 1$$

$$= A_1 + A_1 r \qquad \text{When } n = 2$$

$$= A_1 + A_1 r + A_1 r^2 \qquad \text{When } n = 3$$

$$= A_1 + A_1 r + A_1 r^2 + A_1 r^3 \qquad \text{When } n = 4$$

$$\vdots \qquad \vdots$$

$$= A_1 + A_1 r + A_1 r^2 + \ldots + A_1 r^{n-1} \qquad n \varepsilon N$$

Therefore, if we let $S_n = \sum_{k=1}^{n} A_1 r^{k-1}$, we have:

(1) $S_n = A_1 + A_1 r + A_1 r^2 + \ldots A_1 r^{n-1}$

Multiplying (1) by r we get:

(2) $rS_n = A_1 r + A_1 r^2 + A_1 r^3 + \ldots A_1 r^{n-1} + A_1 r^n$

Subtracting (2) from (1) we get:

$$S_n - rS_n = A_1 - A_1 r^n$$

$$S_n \, (1-r) = A_1 \, (1-r^n)$$

$$S_n = \frac{A_1 \, (1 - r^n)}{(1 - r)}$$

Example 9-5-6

What is the sum of the first 5 terms of the sequence whose terms are: $\{\frac{3}{2}, \frac{3}{8}, \frac{3}{32}, \ldots\}$?

Solution: $r = \frac{1}{4}, A_1 = \frac{3}{2}, n = 5$

$$S_n = \frac{A_1(1 - r^n)}{1-r}$$

$$S_5 = \frac{3/2\ (1 - 1/1024)}{3/4}$$

$$= 2\left(\frac{1023}{1024}\right)$$

$$= \frac{1023}{512}$$

Example 9-5-7

If the 4th term of a geometric sequence is 6 and the 9th term is 1458, find the sum of the first 10 terms.

Solution: $A_4 = 6$ and $A_9 = 1458$

Therefore, $A_4 = A_1r^3 = 6$

$$A_9 = A_1r^8 = 1458$$

$$A_1r^3 = 6 \rightarrow A_1 = \frac{6}{r^3}$$

$$A_1r^8 = \frac{6}{r^3} \cdot r^8 = 1458$$

$$6r^5 = 1458$$

$$r^5 = 243$$

$$r = 3$$

$$A_1 r^3 = 6$$

$$A_1 (3)^3 = 6$$

$$A_1 = \frac{6}{27} = \frac{2}{9}$$

$$S_n = \frac{A_1 (1-r^n)}{1-r}$$

$$S_n = \frac{2/9 \; (1-3^{10})}{1-3}$$

$$S_n = -\frac{1}{9}(1-3^{10}) = 3^8 - \frac{1}{9} = 6560 \frac{8}{9}$$

Exercise 9-5-8

1. Refer to Exercise 9-5-5, problem 1. Use the computer to find the sum of the number of terms indicated.

2. You can be employed to work for a month (30 days) at a pay rate of 1¢ the first day, 2¢ the second day, 4¢ the third day, and so on, (doubling each day). Are you interested in the job?

   What would your pay be on the 30th day of the month? Your total pay for the month?

3. The population of a certain city is now 60,000 and increases by 4% each year. Express this situation by a geometric sequence. What is the nth term?

4. The length of the first swing of a pendulum is 10 in. The length of each succeeding swing is one ninth less than the preceeding one. How long is the 7th swing? The 9th swing?

5. A certain bank pays 4% interest, compounded annually, on deposits. If you deposit $1,000, what will be the balance of your account at the end of six years discounting any further deposits or withdrawls.

6. Find $\sum_{k=5}^{99} r^k$

7. Find n if $\sum_{k=1}^{n} 2^k = 126$

8. Find, if possible, the 1st and 2nd terms of a geometric sequence with 3rd term = -4, 5th term = -1, 8th term = - 1/8.

9. Find all sets of 3 integers in geometric sequence whose product is -216 and the sum of whose squares is 189.

10. The terms between two terms of a geometric sequence are called geometric means. If there is only one term between two terms, the middle term is called the geometric mean between the other two. Insert

    (a) 3 geometric means between 1 and 256,

    (b) 2 geometric means between $\sqrt{5}$ and 5,

    (c) the geometric mean between a^8 and $16b^4$,

    (d) the geometric mean between a and b.

In the exercise above we were able to find the sum of a finite number of terms of a geometric sequence. What is the sum of an infinite number (all) of the terms of a geometric sequence?

Example 9-5-9

Consider $\sum_{n=1}^{\infty} 2n$. The symbol ∞ above the summation sign means that n increases without bound.

$$\sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + 16 + 32 + \ldots$$

As n increases the value of $\sum_{n=1}^{\infty} 2^n$ increases without bound. Therefore, we say that the value of $\sum_{n=1}^{\infty} 2^n$ does not exist. In other words, the sum of all the terms of the sequence $\{(n, 2^n)\}$ does not exist.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

The partial sums are:

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$$

$$\vdots \qquad\qquad \vdots$$

Figure 9-5-11

It appears from Figure 9-5-11 that the value of $\sum_{m=1}^{\infty} \frac{1}{2^n}$ approaches 1 as n increases without bound.

In other words, the sum of all the terms of $\{(n, \frac{1}{2^n})\}$ is 1. We will attempt to verify this by the use of the geometric sum formula $S_n = \frac{A_1(1 - r^n)}{1 - r}$.

In $\{(n, \frac{1}{2^n})\}$, $A_1 = \frac{1}{2}$, $r = \frac{1}{2}$, $n \to \infty$, (increases without bound).

$$S_r = \frac{1/2\ (1 - (1/2)^n)}{1 - 1/2}$$

$$S_n = \frac{1}{2}\ (1 - \frac{1}{2^n})2$$

$$S_n = 1 - \frac{1}{2^n}$$

We showed in Section 9-3 that the limit of the sequence $\{(n, \frac{1}{2^n})\}$ is zero. That is, the value of $\frac{1}{2^n}$ approaches zero as $n \to \infty$

Therefore $S_n = 1 - \frac{1}{2^n} = 1$ when $n \to \infty$.

The difference in Examples 9-5-9 and 9-5-10 above is that for Example 9-5-9 $r > 1$ and for Example 9-5-10 $0 < r < 1$.

When $0 < r < 1$, i.e. $r = \frac{1}{k}$, $K \varepsilon I > 1$,

$$r^n = (\frac{1}{k})^n$$

$$r^n = \frac{1}{k^n}$$

$$r^n = 0 \text{ when } n \to \infty$$

Therefore the formula $S_n = \frac{A_1(1 - r^n)}{1 - r}$ becomes $S_n = \frac{A_1}{1 - r}$.

We will try this new formula on $\{(n, \frac{1}{2^n})\}$.

$$S_n = \frac{1/2}{1 - 1/2}$$

$$S_n = \frac{1/2}{1/2}$$

$S_n = 1$ (The same result we got before)

Exercise 9-5-12

Using the formula $S_k = \frac{A_1}{1-r}$ find:

1. $$\sum_{n=1}^{\infty} (\tfrac{3}{4})^n$$

2. $$\sum_{n\to 1}^{\infty} \tfrac{1}{2}(\tfrac{1}{3})^n$$

3. $$\sum_{n=1}^{\infty} 2(\tfrac{2}{5})^{n-1}$$

4. $$\sum_{n=1}^{\infty} 3^6(\tfrac{1}{3})^n$$

5. The repeating decimal $.\overline{3}$ can be represented as

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \dots ,$$

What is the sum of the terms of the expression?

6. Find the rational number $\frac{p}{q}$ equivalent to each of the following expressions:

(a) $.\overline{6}$ (b) $.\overline{75}$ (c) $1.\overline{14}$ (d) $2.\overline{134}$.

9-6

In Chapter 6 it was stated that the values of the sin function could be calculated for each real x by the use of the following formula:

$$\sin(x) = x^1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \dots$$

Upon examination we note that this is merely a formula for finding the partial sums of the sequence.

$$\{(n, (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!})\}$$

Definition 9-6-1 Factorial

N! (read n factorial) is defined to be the product of $1 \cdot 2 \cdot 3 \cdot 4 \ldots n$, $n \epsilon N$. $0! = 1$.

Example 9-6-2

$$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$$

Exercise 9-6-3

Evaluate:

1. $5!$
2. $\frac{13!}{9!}$
3. $\frac{6!}{(8-4)!}$
4. Write a computer program to evaluate n! for any natural number n.
5. Write a computer program to find the partial sum of the sequence $\{(n, (-1)^{n-1} \frac{x^{2n - 1}}{(2n - 1)!})\}$ for n = 4. Use the program to print out the values of sin(x), $0 \leq x \leq .700$ in intervals of .035. Compare the results with some trigonometric table.
6. In the chapter on logarithms you will be introduced to an irrational number e, which can be calculated to any number of significant digits by either of the formulas

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n \qquad \text{or} \qquad e = \lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{(n - 1)!}$$

Write computer programs to evaluate e by each sequence. Compare the output of the two programs.

Let us now use some of the properties of sequences to approximate the area of the unit circle. Consider the area of a quarter circle.

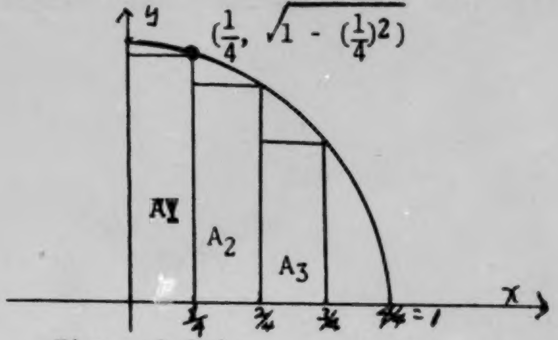


Figure 9-6-4

Dividing the segement $[0,1]$ into 4 equal segements we are able to construct three rectangles with bases equal to 1/4 unit. Since the equation of the unit circle is $x^2 + y^2 = 1$ if follows that $y = \sqrt{1 - x^2}$ for each $0 \le x \le 1$. Any point on the Quarter Circle would be $(x, \sqrt{1 - x^2})$. Therefore, the height of each rectangle is $\sqrt{1 - x^2}$ where x is either the left or right coordinate of the base of each rectangle. Let (K) represent the total area of the three rectangles.

Then $K = A_1 + A_2 + A_3$

$$A_1 = b_1h_1 = \frac{1}{4} \cdot \sqrt{1 - (\frac{1}{4})^2} = \frac{1}{4}\sqrt{\frac{4^2 - 1^2}{4^2}} = \frac{1}{4^2}\sqrt{4^2 - 1^2}$$

$$A_2 = b_2h_2 = \frac{1}{4} \cdot \sqrt{1 - (\frac{2}{4})^2} = \frac{1}{4}\sqrt{\frac{4^2 - 2^2}{4^2}} = \frac{1}{4^2}\sqrt{4^2 - 2^2}$$

$$A_3 = b_3h_3 = \frac{1}{4} \cdot \sqrt{1 - (\frac{3}{4})^2} = \frac{1}{4}\sqrt{\frac{4^2 - 3^2}{4}} = \frac{1}{4^2}\sqrt{4^2 - 3^2}$$

$$\therefore K = \frac{1}{4^2}(\sqrt{4^2 - 1^2} + \sqrt{4^2 - 2^2} + \sqrt{4^2 - 3^2})$$

$$K = \sum_{i=1}^{3} \frac{1}{4^2}\sqrt{4^2 - i^2}$$

The total area of the 3 rectangles certainly is less than the area of the quarter circle.

Now divide the quarter circle as follows

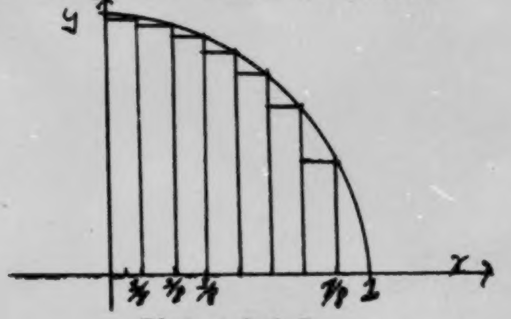


Figure 9-6-5

Now there are 7 rectangles whose total area is less than but nearer the area of the quarter circle.

$$A_1 = \frac{1}{8}\sqrt{1 - (\frac{1}{8})^2} = \frac{1}{8^2}\sqrt{8^2 - 1^2}$$

$$A_2 = \frac{1}{8}\sqrt{1 - (\frac{2}{8})^2} = \frac{1}{8^2}\sqrt{8^2 - 2^2}$$

$$A_3 = \frac{1}{8}\sqrt{1 - (\frac{3}{8})^2} = \frac{1}{8^2}\sqrt{8^2 - 3^2}$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$A_7 = \frac{1}{8}\sqrt{1 - (\frac{7}{8})^2} = \frac{1}{8^2}\sqrt{8^2 - 7^2}$$

$$K = A_1 + A_2 + A_3 + \ldots + A_7$$

$$K = \frac{1}{8^2}\left(\sqrt{8^2 - 1^2} + \sqrt{8^2 - 2^2} + \sqrt{8^2 - 3^2} + \ldots + \sqrt{8^2 - 7^2}\right)$$

$$K = \sum_{i=1}^{7} \frac{1}{8^2}\sqrt{8^2 - i^2}$$

If the interval from 0 to 1 were divided into 16 equal segements we could construct 15 rectangles each having width 1/16 unit. The total area of the rectangles would be closer to the area of the quarter circle than the total area of the rectangles with bases 1/8 unit.

It should now be evident that the area of the 15 rectangles would be:

$$K = \frac{1}{16}\sqrt{16^2 - 1^2} + \frac{1}{16}\sqrt{16^2 - 2^2} + \frac{1}{16}\sqrt{16^2 - 3^2} + \ldots + \frac{1}{16}\sqrt{16^2 - 15^2}$$

$$K = \sum_{i=1}^{15} \frac{1}{16^2}\sqrt{16^2 - i^2}$$

Obviously as we increase the number of subdivisions of the quarter circle the total area of the rectangles would approach the area of the quarter circle. In other words if we have n subdivisions the sum of the area of n-1 rectangles formed approaches the area of the quarter circle as n increases.

Exercise 9-6-6

1. Write the generalization for finding the sum of the n-1 rectangles found by dividing a quarter of the unit circle into n equal intervals.

2. Use the generalization of problem 1 to write a computer program that will print out the approximate area of the unit circle for any n. Remember if K is the area of the quarter circle then 4K is the area of the unit circle.

   Use the program to print out the approximate area of the unit circle when the interval 0 to 1 is divided into n equal intervals for:

   (a) n = 10, 20, 30, ..., 100

   (b) n = 100, 200, 300, ..., 1000

   (c) n = 1000, 2000, 3000, ..., 10000

3. Does it appear that the partial sum sequence above is converging to some number familiar to you? If so, what number?

4. By using the area of the quarter circle and multiplying by 4 to approximate the area of the unit circle, we were actually summing the rectangles as illustrated on the following page?

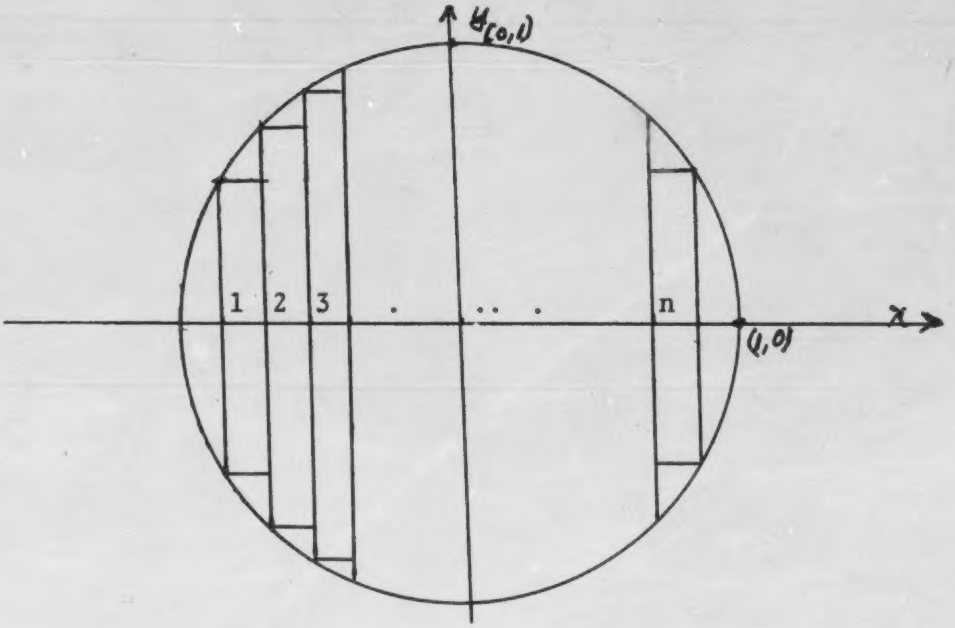


Figure 9-6-7

If n is large, then the total area of the rectangles approaches the area of the circle as its limit. Why would that limit be π as conjectured in 3 above?

Chapter 10

Exponential and Logarithmic Functions

10-1 Introduction

In this chapter we will review and extend the properties of exponents along with introducing exponential and logarithmic functions.

10-2 Natural Number Exponents and Logarithms

Let us begin by reviewing work we have already done with natural number exponents. Before beginning Exercise 10-2-2, consider the following definition.

Definition 10-2-1 Natural Number Exponents

$\forall x,\ x^1 = x$ and $x^{n+1} = x \cdot x^n,\ n \in N$

This is a recursive definition and may be informally interpreted as saying

$$x^n = \overbrace{x \cdot x \cdot x \ldots x}^{n \text{ times}}.$$

Exercise 10-2-2

1. Simplify each of the following expressions.

   a. $3^2 \cdot 3^5$ b. $7^4 \cdot 7^9$

2. Complete the equation $a^x \cdot a^y =$ ________, where $a \in R$, $x \in N$, $y \in N$.

3. Simplify each of the following expressions.

   a. $\frac{3^7}{3^4}$ b. $\frac{7^9}{7^4}$

4. Complete the equation $\frac{a^x}{a^y} =$ ________, where $a \in R$, $a \neq 0$, $x \in N$, $y \in N$, $x > y$.

5. Write equivalent exponential expressions for the following.

   a. $(3^2)^7$ b. $(3^7)^2$

   c. $(7^4)^5$ d. $(7^5)^4$

6. Complete the equations $(a^x)^y = (a^y)$_____ = _____, where $a \in R$, $x \in N$, $y \in N$.

7. Write equivalent exponential expressions for the following.

   a. $(\frac{3^2}{5^2})$ b. $(\frac{4}{5})^3$

   c. $(\frac{2^2}{3})^5$ d. $(\frac{3^6}{7^3})$

8. Complete the equation $(\frac{a}{b})^x =$ __________, where $a \in R$, $b \in R$, $b \neq 0$, $x \in N$.

9. Simplify the following expressions using the principles for natural number exponents reviewed in problems 1-8. Consider x, y and z non-zero real numbers and a, b and $c \in N$.

   a. $5^2 \cdot 5^3 \cdot 5$ b. $(9^2)^5 \cdot 9^3$

   c. $(\frac{7^9}{7^4})(7^3)$ d. $(\frac{4^5}{4^3})^3$

   e. $2x(2^3 x^2)$ f. $(3x^2y^3)(3^2 x y^2)$

   g. $\frac{x^{2a}}{x^a}$ h. $(\frac{x^6 \quad y^4}{z^2})(\frac{z^3}{x^3 \quad y^2})$

   i. $\dfrac{\frac{-30x^2y}{85x}}{\frac{1}{x} - \frac{1}{y}}$ j. $\left(-\dfrac{\frac{90(x^2y)^3}{16x^3}}{\frac{180 x y^3}{81}}\right)^2$

10. Consider the following axiom relating to exponents: $\forall x \quad x^a = x^b \rightarrow a=b$, $a, b \in N$.
    Using the axiom, solve the following equations for x.

    a. $15^3 = 15^{x+2}$

    b. $(12^x)(12^2) = 12^9$

    c. $5^{x^2 - 2x} = 5^3$

    d. $\frac{7^6}{7^x} = 7^x$

Our initial discussion of the exponential function will deal with natural number exponents. Let's begin by observing whether the relation $\{(x,y)|y = 2^x, x\epsilon N\}$ is a function.

Example 10-2-3

Write a computer program which will produce the first ten terms of the relation $\{(x,y)|y = 2^x, x\epsilon N\}$.

```
10  FOR X = 1 TO 10
20  LET Y = 2 ↑ X
30  PRINT X,Y
40  NEXT X
50  END
```

The output of the program, in the form of a table, appears in Figure 10-2-4.

x	1	2	3	4	5	6	7	8	9	10
y	2	4	8	16	32	64	128	256	512	1024

Figure 10-2-4

This relation is a function and this finite table is a subset of the sequence $\{(x,2^x)\}$. The table consists of two sequences. The first of which, $\{(x,x)\}_1^{10}$, is arithmetic and the second $\{(x,2^x)\}_1^{10}$ is geometric. This one-to-one correspondence between as arithmetic and a geometric sequence is the essence of a system of logarithms. Figure 10-2-4 is a subset of the logarithm system, base two.

Definition 10-2-5 Logarithm

The logarithm, to the base a, of x is the exponent or the power to which a is raised to obtain x.

One of the uses of a system of logarithms is to simplify the work of finding products and quotients. Figure 10-2-4 can be helpful in finding products of particular numbers as in the next example.

Example 10-2-6

Find the product of $16 \cdot 32$.

We know that:

$$\begin{array}{r} 32 \\ \times\ 16 \\ \hline 192 \\ 32 \\ \hline 512 \end{array}$$

But from the table we see that

16 corresponds to 4

and 32 corresponds to 5.

Now $4 + 5 = 9$ and 9 corresponds to our known product, 512. In this case, 4, 5 and 9 are the logarithms base 2, of 16, 32 and 512 respectively.

Figure 10-2-7 is a table similar to Figure 10-2-4, only it illustrates the sequence $\{(x, 3^x)\}_1^{10}$.

x	1	2	3	4	5	6	7	8	9	10
y	3	9	27	81	243	729	2187	6561	19683	59049

Figure 10-2-7

Example 10-2-8

Find the product of 27 and 243.

$$\begin{array}{rlrl} 243 & \text{corresponds to} & 5 & \text{(the logarithm base 3 of 243)} \\ \times\ 27 & \text{corresponds to} & \underline{+\ 3} & \text{(the logarithm base 3 of 27)} \\ \hline 1701 & & & \\ 486 & & & \\ \hline 6561 & \text{corresponds to} & 8 & \text{(the logarithm base 3 of 6561)} \end{array}$$

Example 10-2-9

Perform the division, $19683 \div 27$.

$$\begin{array}{ccccc} 19683 & \div & 27 & = & 729 \\ \updownarrow & & \updownarrow & & \updownarrow \\ 9 & - & 3 & = & 6 \end{array}$$

(logarithm base 3 of 19683) (logarithm base 3 of 27) (logarithm base 3 of 729)

Note that in Example 10-2-6, logarithm of 512 is 9 and that from Example 10-2-9, logarithm of 19683 is 9. This illustrates the need for a notational system to distinguish between different systems of logarithms. We distinguish between the different systems by using a subscript which denotes the base in which we are working. In this case, $\log_2 512 = 9 = \log_3 19689$.

Example 10-2-10

Since $3^2 = 9$, $\log_3 9 = 2$.

$\log_3 9 = 2$ is read as follows:

The logarithm to the base 3 of 9 is 2. Notice that the logarithm, 2, is the exponent of the base, 3, which results in 9.

Exercise 10-2-11

1. Complete the following statements.

   a. Since $2^4 = 16$, $\log_2 16 =$ _____. The logarithm to the base, _____, of 16 is _____.

   b. Since $4^3 = 64$, $\log_4 64 =$ _____. The logarithm to the base, 4, of _____ is _____.

   c. Since $10^1 = 10$, $\log_{10} 10 =$ _____. The logarithm to the base, _____, of _____ is _____.

2. Complete the following tables.

a. $y = 2^x$

x	y
1	2
3	8
4	16
7	128
⋮	

b. inverse of $y = 2^x$

x	y
⋮	

c. $x = 2^y$

x	y
	1
	3
	4
	7
⋮	

d. $y = \log_2 x$

x	y
2	
8	
16	
128	
⋮	

3. State at least two ways of expressing the inverse of $y = 5^x$.

The following tables, Figure 10-2-12, consist of powers of 2,3,4,5,7, and 9. Use them as an aid in solving the problems in Exercise 10-2-13.

Figure 10-2-12

x	2	4	8	16	32	64	128	256	512	1024
$\log_2 x$	1	2	3	4	5	6	7	8	9	10
x	3	9	27	81	243	729	2187	6561	19683	59049
$\log_3 x$	1	2	3	4	5	6	7	8	9	10
x	4	16	64	256	1024	4096	16384	65536	262144	1048576
$\log_4 x$	1	2	3	4	5	6	7	8	9	10
x	5	25	125	625	3125	15625	78125	390625	1953125	9765625
$\log_5 x$	1	2	3	4	5	6	7	8	9	10
x	7	49	343	2401	16807	117649	823543	5764801	40353607	282475249
$\log_7 x$	1	2	3	4	5	6	7	8	9	10
x	9	81	729	6561	59049	531441	4782969	43046721	387420489	3486784401
$\log_9 x$	1	2	3	4	5	6	7	8	9	10

Exercise 10-2-13

1. Using tables from Figure 10-2-12 simplify the following expressions.

   a. $\log_7 (343 \cdot 2401)$

   b. $\log_7 343 + \log_7 2401$

   c. $\log_4 64 + \log_4 1024$

   d. $\log_4 (64 \cdot 1024)$

2. Complete the following equations where $0 < a \neq 1$ and x and y are natural number powers of a.

   a. $\log_a x + \log_a y =$ ______________________

   b. $\log_a (x \cdot y) =$ ______________________

It seems from our experience that the generalizations above, are true. After introducing a formal definition of logarithms we will be able to show that they are theorems.

3. Simplify the following expressions.

   a. $\log_5 3125 - \log_5 125$

   b. $\log_5 (\frac{3125}{125})$

   c. $\log_9 4782969 - \log_9 6561$

   d. $\log_9 (\frac{4782969}{6561})$

4. Complete the following equations, where $0 < a \neq 1$, x and y natural number powers of a.

   a. $\log_a (\frac{x}{y}) =$ ____________________

   b. $\log_a x - \log_a y =$ ____________________

   Proof of these generalizations will follow the formal definition of logarithms.

5. Simplify the following expressions.

   a. $\log_2 (4 \cdot 4 \cdot 4)$

   b. $\log_2 (4)^3$

   c. $3 \log_2 4$

   d. $\log_7 (343)^2$

   e. $2 \log_7 (343)$

6. Complete the following equations, where $0 > a \neq 1$, $b \in N$, x a natural number power of a.

   a. $\log_a (x^b) =$ ______________________

   b. $b \log_a x =$ ______________________

   Proof of these generalizations will follow the formal definition of logarithm.

7. Using as an axiom, $\log_a x = \log_a y \Rightarrow x = y, \forall a > 0$ and $a \neq 1$, x and y natural number powers of a, solve the following problems for x, using the tables from Figure 10-2-11.

   a. $\log_3 81 + \log_3 243 = \log_3 x$

   b. $\log_3 (81 \cdot 243) = \log_3 x$

   c. $\log_3 19683 - \log_3 81 = \log_3 x$

   d. $\log_3 (\frac{19683}{81}) = \log_3 x$

   e. $\log_5 125 + \log_5 x = \log_5 (390625)$

   f. $\log_5 (125 \cdot x) = 8$

   g. $\log_5 (390625) - \log_5 x = \log_5 5^2(5)$

   h. $\log_5 (\frac{390625}{x}) = \log_5 5^3$

   i. $\log_9 x + \log_9 (729) = \log_9 (9^4 \cdot 9^3)$

   j. $\log_9 (729x) = \log_9 (4{,}782{,}969)$

   k. $\log_9 (x) - \log_9 (6561) = \log_9 (9^3)$

   l. $\log_9 (\frac{x}{6561}) = 3$

10-3 Integral Exponents

Having reviewed the properties of natural number exponents and discovered many properties of logarithms, we are now prepared to consider integral exponents and the behavior of logarithms in this expanded domain.

Exercise 10-3-1

Complete the missing values in the following tables, Figure 10-3-2 and 10-3-3.

-10								-2	-1		1	2	3	4	5
							$\frac{1}{8}$			1	2	4	8	16	32

Figure 10-3-2

-10							-3				1	2	3	4	81
								$\frac{1}{9}$		1	3	9	27	81	243

Figure 10-3-3

In order to properly assess the value of the above tables to our study of exponents and logarithm systems we make the following definitions.

Definition 10-3-4 Non-positive, Integral Exponents

$Vx,\ x \neq 0 \quad x^{\circ} = 1$

$Vx,\ x \neq 0,\ Va,\ a \varepsilon I,\ a \neq 0,\ x^{-a} = \frac{1}{x^{a}}$

To review additional properties of exponents and further understand systems of logarithms, consider the tables in Figure 10-3-5 and use them in solving the problems from Exercise 10-3-6.

-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$\frac{1}{1024}$	$\frac{1}{512}$	$\frac{1}{256}$	$\frac{1}{128}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	256	512	1024

-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$\frac{1}{59049}$	$\frac{1}{19683}$	$\frac{1}{6561}$	$\frac{1}{2187}$	$\frac{1}{729}$	$\frac{1}{243}$	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81	243	729	2187	6561	19683	59049

-5	-4	-3	-2	-1	0	1	2	3	4	5
.00001	.0001	.001	.01	.1	1	10	100	1000	10000	100000

Figure 10-3-5

Exercise 10-3-6

1. Using Figure 10-3-5 simplify the following.

   a. $3^{-2} \cdot 3^{-3}$ b. $2^{-7} \cdot 2^{2}$

   c. $\frac{2^5}{2^7}$ d. $\frac{3^{-1}}{3^{-6}}$

   e. $\frac{2^4}{2^{-9}}$ f. $\frac{10^5}{10^{-2}}$

   g. $(10^{-2})^3$ h. $(10^3)^{-2}$

   i. $(\frac{2^{-2}}{3})^4$ j. $(\frac{10}{3})^{-3}$

2. The generalizations from Exercise 10-2-2 are $\forall a\ \forall b$, $a \neq 0 \neq b$ and $\forall x$, $\forall y$, x, $y \in N$,

$$a^x \cdot a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{x \cdot y}$$

$$\frac{a^x}{b^x} = (\frac{a}{b})^x$$

   If the restrictions on a,b,x, and y are changed to $\forall a\ \forall b$, $a \neq 0 \neq b$, $\forall x, \forall y, x, y \in I$, are any of these generalizations false?

3. Using Figure 10-3-5 and previous knowledge of logarithms simplify the logarithmic form of the following expressions.

   a. $\log_3 (\frac{1}{27} \cdot \frac{1}{243})$ b. $\log_3 (\frac{1}{27}) + \log_3 \frac{1}{243}$

   c. $\log_2 (32 \cdot \frac{1}{16})$ d. $\log_{10} (1000) + \log_{10} (.01)$

   e. $\log_3 (\frac{243}{1/9})$ f. $\log_3 243 - \log_3 (\frac{1}{9})$

   g. $\log_2 (\frac{1/64}{1/16})$ h. $\log_2 (\frac{1}{128}) - \log_2 (\frac{1}{4})$

   i. $\log_{10} (100)^2$ j. $3 \log_3 (\frac{1}{9})$

4. In Exercise 10-2-13 the following generalizations were formed relating to logarithms, Va, VbϵN, $a \neq 1$ and x and y natural numbers.

$$\log_a (x \cdot y) = \log_a x + \log_a y$$
$$\log_a \left(\frac{x}{y}\right), y \neq 0 = \log_a z - \log_a y$$
$$\log_a x^b = b \log_a x$$

If the restriction on x and y is changed to integral powers of a and b, are these generalizations still true?

5. Simplify the following statements.

a. $2^3 \cdot 2^{-3}$ b. $10^2 \cdot 10^{-2}$

c. $\log_3 \frac{1}{27} + \log_3 27$ d. $\log_{10} (\frac{1}{1000} \cdot 1000)$

6. Complete the following equations.

a. Vx, $x \neq 0$, Va, aϵI, $\frac{1}{x^a}$ = ________________

b. Vx, $x \neq 0$, x^0 = ________________

c. Va,aϵN, $a > 1$, V$x \neq 0$ an integral power of a, $\log_a \frac{1}{x}$ = __________

7. Simplify each of the following expressions where x is a real number $\neq 0$, y a positive real number, $y \neq 1$, a an integer, and $0 < b \neq 1$.

a. $x^a \cdot x^{-a}$ b. $\frac{x^{a-1}}{x^{a-b}}$

c. $\frac{\cos^2\theta}{\sin^{-2}\theta} \left(\frac{\sin^4 \theta}{(\sin \theta)(\cos^5 \theta)}\right)^2$ d. $\frac{a^{-2} - b^{-2}}{a^{-1} + b^{-1}}$

e. $\log_b x + \log_b y$ f. $\log_b \frac{1}{y}$

g. $\frac{\log_b 2x + \log_b 3}{\log_b 5x - \log_b x}$ h. $\log_b \left(\frac{x \cdot 3y}{(y^3)(x)}\right)$

i. $\log_b (b)^{-a}$

8. Solve each of the following equations for x.

a. $x = \log_9 81 + \log_9 59049$

b. $a^x \cdot a^x = a^{9x}$

c. $b^{-x} \cdot b^{x+2} = b^{x-4}$

d. $\log_4 \frac{1}{16} = x$

e. $(b^x)^{x+4} = b^{12}$

f. $\log_3 27 = x$

g. $\log_{27} x = -5$

h. $\log_7 \frac{x}{2401} = -3$

i. $\log_x \frac{81}{(81)^a} = -2$

j. $\log_5 125 = \log_5 x + \log_5 (\frac{25}{625})$

k. $\log_{10} 10^x = 27 \log_{10} 10$

l. $\log_5 x^5 - \log_5 x^7 = -4$

10-4 Rational Number Exponents

In this section we want to increase our understanding of exponents and systems of logarithms. First we will give the reader some basis for understanding what a rational number exponents is. We will then approximate the value of various bases raised to powers which are rational numbers.

Let us center our discussion on the one-to-one correspondence between $\{(x, 10^x)\}$ and $\{(x, x)\}$. Figure 10-4-1 illustrates a finite subset of the correspondence mentioned. While the following discussion lends itself to any base, ten is used for purposes of discussion because of its use as a computational logarithm system. Logarithms, base 10, are usually referred to as Common Logarithms and written log x.

log x	-4	-3	-2	-1	0	1	2	3
x	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10	10	10	10

Figure 10-4-1

The table for the base 10 system in Figure 10-4-1 may be expanded by inserting the arithmetic mean between successive terms of the arithmetic sequence $\{-4,-3,-2,-1,0,1,2,3\}$. The values which correspond to these new terms are the positive geometric means which lie between successive terms of the geometric sequence $\{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^0, 10^1, 10^2, 10^3\}$. This means that the refined table will consist of an arithmetic sequence whose common difference is one-half and a corresponding geometric sequence whose common ratio is the positive square root of the original ratio, in this example, $\sqrt{10}$.

Example 10-4-2

log x	x	Equivalent x	Approximate x
.	.	.	.
.	.	.	.
.	.	.	.
-4	10^{-4}	$\frac{1}{10^4}$	0.0001
$-\frac{7}{2}$	$10^{-7/2}$	$\frac{1}{\sqrt{10^7}}$	0.000316228
-3	10^{-3}	$\frac{1}{10^3}$	0.001

Continued on Page 10-15

$-\frac{5}{2}$	$10^{-5/2}$	$\frac{1}{\sqrt{10^5}}$	0.00316228
-2	10^{-2}	$\frac{1}{10^2}$	0.01
$-\frac{3}{2}$	$10^{-3/2}$	$\frac{1}{\sqrt{10^3}}$	0.0316228
-1	10^{-1}	$\frac{1}{10}$	0.1
$-\frac{1}{2}$	$10^{-1/2}$	$\frac{1}{\sqrt{10}}$	0.316228
0	10^0	1	1.0
$\frac{1}{2}$	$10^{1/2}$	$\sqrt{10}$	3.16228
1	10^1	10	10.0
$\frac{3}{2}$	$10^{3/2}$	$\sqrt{10^3}$	31.6228
2	10^2	10^2	100.0
$\frac{5}{2}$	$10^{5/2}$	$\sqrt{10^5}$	316.228
3	10^3	10^3	1000.0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 10-4-3

Let us first verify that the use of the rational number exponents, in Figure 10-4-3, conform to the hypothesis about logarithms which states

$$\log_a x + \log_a y = \log_a (x \cdot y).$$

Example 10-4-4

Evaluate $\log 10^{-7/2} + \log 10^{5/2}$ using Figure 10-4-3.

$$\log 10^{-7/2} = -\frac{7}{2}$$

$$\log 10^{5/2} = \frac{5}{2}$$

$$\therefore \log 10^{-7/2} + \log 10^{5/2} = -\frac{7}{2} + \frac{5}{2} = -1$$

Since $-1 = \log 10^{-1}$

We may state $\log 10^{-7/2} + \log 10^{5/2} = \log 10^{-1}$

Observe that the sum of the exponents on the left, $-\frac{7}{2} + \frac{5}{2}$, is equal to the exponent on the right, -1.

Example 10-4-5

Evaluate $\log 10^{-1/2} + \log 10^{-2}$ using Figure 10-4-3

$$\log 10^{-1/2} = -\frac{1}{2}$$

$$\log 10^{-2} = -2$$

$$\therefore \log 10^{-1/2} + \log 10^{-2} = -\frac{1}{2} + -2 = -\frac{5}{2}$$

and $-\frac{5}{2} = \log 10^{-5/2}$

Hence $\log 10^{-1/2} + \log 10^{-2} = \log 10^{-5/2}$

Again observe that the sum of the exponents $-\frac{1}{2} + -2$ is $-\frac{5}{2}$.

On the basis of Examples 10-4-4 and 10-4-5, we make the following definition.

Definition 10-4-6 Rational Number Exponents

$\forall a, \forall x,y$ rational numbers, $a^x \cdot a^y = a^{x+y}$

Now that we have the above definition, 10-4-6, we can expand the work done in Example 10-4-4 and 10-4-5 as follows.

$$\log 10^{-7/2} + \log 10^{5/2} = \log 10^{-1} = \log (10^{-7/2} \cdot 10^{5/2})$$

and

$$\log 10^{-1/2} + \log 10^{-2} = \log 10^{-5/2} = \log (10^{-1/2} \cdot 10^{-2})$$

It appears that for rational numbers, $\log_a x + \log_a y = \log_a(x \cdot y)$.

Exercise 10-4-7

1. Expand the $\log_{10} x$ table found in Figure 10-4-3 from $x = 10^{-10}$ to 10^{10} with arithmetic and geometric means between integral values of $\log_{10} x$. Approximations for x, while not necessary for this exercise, will be informative in succeeding sections of this chapter.

2. On the same Cartisian coordinate system graph the table constructed in problem one, where $y = \log_{10} x$ and its inverse $y = 10^x$. What is the relationship between these two functions?

3. Evaluate each of the following:

   a. $\log 10^{3/2} + \log 10^{7/2}$
   b. $\log(10^{3/2} \cdot 10^{7/2})$
   c. $\log 10^{-7/2} + \log 10^{2}$
   d. $\log (10^{-7/2} \cdot 10^{2})$
   e. $\log (10^{1/2})^{3}$
   f. $3 \log 10^{1/2}$
   g. $-5 \log 10^{-3/2}$
   h. $\log (10^{-3/2})^{-5}$
   i. $\log 10^{5/2} - \log 10^{3}$
   j. $\log \left(\frac{10^{5/2}}{10^{3}}\right)$
   k. $\log 10^{-19/2} - \log 10^{-17/2}$
   l. $\log \left(\frac{10^{-19/2}}{10^{-17/2}}\right)$
   m. $\log \frac{1}{10^{-3/2}}$
   n. $-\log 10^{-3/2}$
   o. $\log 10^{3/2}$
   p. $\log(10^{-3/2})^{-1}$

4. Use Definition 10-2-5 to assist you in completing each of the following statements.

   a. Since $10^{1/2} = \sqrt{10}$, $\log \sqrt{10} =$ _____. The logarithm to the base, 10, of $\sqrt{10}$ is 1/2.

   b. Since $10^{-19/2} = \frac{1}{\sqrt{10^{19}}}$, $\log \frac{1}{\sqrt{10^{19}}}$ = _____. The logarithm to the base, _____, of _____ is $-\frac{19}{2}$.

c. If $2^{3/2} = \sqrt{2^3}$, $\log_2 \sqrt{2^3} =$ _____. The logarithm to the base, _____, of _____ is _____.

d. If $3^{-7/2} = \frac{1}{\sqrt{3^7}}$, $\log_3$ _____ = _____. The logarithm to the base, 3, of _____ is _____.

e. If $10^{5/3} = \sqrt[3]{10^5}$, $\log_{10} \sqrt[3]{10^5} =$ _____. The logarithm to the base _____, of _____ is _____.

Let us again look at a portion of the $\log_{10}x$ table and a further refinement.

Example 10-4-8

$\log_{10}x$	0	1
x	1	10

In the table above, find the two arithmetic means between 0 and 1, and the two corresponding geometric means between 1 and 10;

We know that, to find the common difference, d, of the arithmetic sequence

$$d = \frac{0+1}{3} = \frac{1}{3}$$

and the common ratio, r, of the geometric sequence

$$10 = 1 \cdot r^3 \qquad r = \sqrt[3]{10}.$$

Therefore, we now have

$\log_{10}x$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
x	1	$\sqrt[3]{10}$	$(\sqrt[3]{10})^2$	10

However, if our tables are consistant

$$\log_{10} x = \frac{1}{3} \rightarrow x = 10^{1/3}$$

and

$$\log_{10} x = \frac{2}{3} \rightarrow x = 10^{2/3}$$

which means that

$$10^{1/3} = \sqrt[3]{10}$$

$$10^{2/3} = (\sqrt[3]{10})^2$$

and our table may be written

$\log_{10} x$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
x	1	$10^{1/3}$	$10^{2/3}$	10

This example, 10-4-8 suggests the following definition.

Definition 10-4-9 Rational Exponents

Vx, x a rational number $a^x = a^{m/n} = \sqrt[n]{a^m}$

Vx, x a rational number $a^{-x} = \frac{1}{a^x}$, $a \neq 0$

Exercise 10-4-10

1. Change each of the following to their simplest exponent form where all exponents are positive. Consider x, y and z positive rational numbers.

a. $\frac{1}{10^{-3/4}}$

b. $\frac{1}{2^{-1} + 3^{0/7}}$

c. $\frac{a^{-1/2}}{a^{-1/2}}$

d. $(\frac{a}{b})^{-x}$

e. $\frac{r}{s^{-y}}$

f. $\frac{cd^{-x}}{4a^{-y}}$

g. $\frac{a^x b^y}{a^{-2} b^{-3/4}}$

h. $(a^x b^y)^{-z}$

i. $a^{-x} + b^{-y}$

j. $\frac{a^{-x} + b^{-y}}{(cd)^{-z}}$

k. $\frac{a^{-x} + b^{-y}}{a^{-x} + b^{-y}}$

l. $\frac{a^{-x} + b^{-y}}{a^{-(x+2)} + b^{-(y+2)}}$

2. Express the common logarithms of each of the following expressions in terms of the logarithms of the letters involved as in the following example:

$\frac{p(\sqrt[3]{Q})^2}{r}$, log $\frac{P(\sqrt[3]{Q})^2}{r}$ = log P + $\frac{2}{3}$ log Q - log r

a. $\frac{Q}{p^2R^3}$ b. $\sqrt{\frac{PQ}{R}}$

c. $\frac{\sqrt{PQ}}{R}$ d. $\sqrt[3]{\frac{P^2Q}{R}}$

e. $(\frac{1}{P\sqrt{Q}})^{2/5}$ f. $\frac{1}{2}(\sqrt{\frac{Q}{R^3}})$

3. Solve for x in each of the following equations.

a. $2^{2x + 6} = 32$

b. $9^{2x/3} = 27$

c. $10^{2x^2 + 4} = 10^x \cdot 10^7$

d. $\log_9 x + \log_9 27 = \log_9 81 - \log_9 2$

e. $\log_2 16 - \log_2 x = -\frac{1}{2}$

f. $\log_{10} 13 + 2 \log_{10} 6 - \frac{3}{2} \log_{10} 4 = \log_{10} x$

In our previous work in systems of logarithms and exponents we found that the logarithm, to a given base, of a number, a, is the exponent to which the base must be raised to obtain the number a. This leads to the following interpretation of the equation $\log_{10} x = 1/16$; since $\log_{10} x = 1/16$, 1/16 is the power 10 must be raised in order to obtain the number x. The question the authors ask now is "what do we mean by $10^{1/16}$ since 1/16 is not an integer"? More generally one might ask "what is meant by x^a when a is not an integer?"

In looking for answers to these questions, one should recall that for all a and for rational x and y

$$a^x \cdot a^y = a^{x+y}$$

$$(a^x)y = a^x \cdot y$$

and

$$Va,\ a>0,\ a^{1/2} = \sqrt{a}$$

Since we know that

$$Va,\ a>0,\ a^{1/2} = \sqrt{a}$$

we seek an interpretation for $a^{1/4}$.

Since $a^{1/4} = a^{1/2 \cdot 1/2} = (a^{1/2})^{1/2}$

we find $a^{1/4} = (\sqrt{a})^{1/2} = \sqrt{\sqrt{a}}$

Likewise, $a^{1/8} = a^{1/4 \cdot 1/2} = (a^{1/4})^{1/2}$

or $a^{1/8} = (\sqrt{\sqrt{a}})^{1/2} = \sqrt{\sqrt{\sqrt{a}}}$

Finally, $a^{1/16} = (\sqrt{\sqrt{\sqrt{a}}})^{1/2} = \sqrt{\sqrt{\sqrt{\sqrt{a}}}}$

Hence, $10^{1/16} = \sqrt{\sqrt{\sqrt{\sqrt{10}}}}$

A similar development may be used to illustrate that $a^{1/9} = \sqrt[3]{\sqrt[3]{a}}$. where $a^{1/3}$ is defined as $\sqrt[3]{a}$. It is true that $\forall a$, $a>0$, $a^{1/x^n} = \sqrt{\sqrt{\dots \sqrt{a}}}$. We will take this fact only as information because presently we desire to evaluate the approximation of bases raised to rational number exponents power of $1/2^n$ achieve the accuracy in our approximation we desire.

Example 10-4-11

Evaluate $10^{.75}$ as a rational exponent compared to its radical form.

$$
\begin{aligned}
\text{Since } 10^{.75} &= 10^{.5 + .25} \\
&= 10^{.5} \cdot 10^{.25} \\
&= 10^{1/2} \cdot 10^{1/4} \\
&= \sqrt{10} \cdot \sqrt{\sqrt{10}}
\end{aligned}
$$

Finally $10^{.75} = \sqrt{10 \cdot \sqrt{10}}$

Now $.75 = \frac{3}{4}$

Therefore $10^{.75} = 10^{3/4} = \sqrt{10\sqrt{10}}$

Example 10-4-12

Evaluate $10^{.37501}$ as a rational exponent by considering its radical form.

Now, $.25 < .37501$, therefore, $10^{.37501} = 10^{.25} \cdot 10^{.12501}$

and $.125 < .12501$ which means $10^{.12501} \cdot 10^{.125} \cdot 10^{.00001}$

Since $10^{.00001}$ is so small lets approximate the exponent to zero. That is, $10^{.00001} \approx 10^{\circ} = 1$.

So, $10^{.37501} \approx 10^{.25} \cdot 10^{.125} \cdot 1$

$$10^{.37501} \approx \sqrt{\sqrt{10\sqrt{10}}}$$

Exercise 10-4-13

In problem 1-10, find the best radical approximation of the rational power of the base. Consider 10^x for $x < 0.001$ approximately equivalent to 1.

1. $10^{.5}$
2. $10^{.25}$
3. $10^{.875}$
4. $10^{1.375}$
5. $10^{.5625}$
6. $10^{4.837951}$
7. $10^{2.321904}$
8. $10^{3.00149}$
9. $2^{2.1432}$
10. $4^{1.0716}$
11. Construct a computer program which will approximate the values of the numbers in problem 1-10 to the same degree of accuracy specified.

10-5 Real Number Exponents

At this point in our development we have considered different exponent bases with the objective of understanding the domain of the exponent. In the last section we also found that we may approximate the value of a^x where a is a non-zero real number and x is a rational number. In a similar manner we may extend the domain of the exponent to include the irrational numbers.

Example 10-5-1

Evaluate 10^π

$$\begin{aligned} 10^\pi &\approx 10^{3.14159} \\ &\approx 10^3 \cdot 10^{.125} \cdot 10^{.015625} \cdot 10^{.000165} \\ &\approx 1000 \cdot \sqrt{\sqrt{\sqrt{10}}} \cdot \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{10}}}}}} \cdot 1 \text{ where } 10^{.000165} \approx 1 \\ &\approx 1000 \cdot \sqrt{\sqrt{\sqrt{10\sqrt{\sqrt{\sqrt{10}}}}}} \end{aligned}$$

The previous example is only one of many which could have been used to verify that the exponential function has as its domain the set of real numbers.

Definition 10-5-2 The Exponential Function

$\forall a > 0$, the function $\{(x,y) | y = a^x, x \epsilon R\}$ is the exponential function, base a.

For $0 < a$ the exponential function is continuous. If $0 < a < 1$ the function is strictly decreasing, if $a = 1$ the function is a constant function, and if $a > 1$ the function is strictly increasing.

Definition 10-5-3 The Logarithm Function

The inverse of the exponential function $\{(x,y) | y = a^x,\}$ $a \neq 1$ is $\{(x,y) | x = a^y, x \epsilon 0\}$, $0 < a \neq 1$, and may be written $\{(x,y) | y = \log_a x, x \epsilon R > 0\}$ $0 < a \neq 1$.

We now have presented the formal definition of the exponential and logarithm function. These definitions will now allow us to prove some of the generalizations we made previously about logarithms.

Theorem 10-5-4

$\forall 0 < a \neq 1, \forall x > 0, \forall y > 0, \log_a(x \cdot y) = \log_a x + \log_a y$

proof: let $p = \log_a x$ and $q = \log_a y$

Then by definition of logarithms,

$a^p = x$ and $a^q = y$

and $a^p \cdot a^q = x \cdot y$

or $a^{p+q} = x \cdot y$

But, $a^{p+q} = x \cdot y \rightarrow \log_a(x \cdot y) = p + q$

where $p = \log_a x$ and $q = \log_a y$

Therefore, $\log_a(x \cdot y) = \log_a x + \log_a y$

Exercise 10-5-5

1. Prove the following theorems of logarithms

   a. $\forall 0 < a \neq 1, \forall x > 0, \forall_n, \log_a x^n = n \log_a x$

   b. $\forall 0 < a \neq 1, \forall x > 0, \forall y > 0, \log_a \frac{x}{y} = \log_a x - \log_a y$

2. Graph, on the same coordinate axis, the following functions.

   $\{(x,y) \mid y = 2^x\}$ and $\{(x,y) \mid y = \log_2 x\}$

3. What is the relationship between the functions in problem 2?

4. Graph, on the same coordinate axis, the following functions.

   $\{(x,y) \mid y = (\frac{1}{2})^x\}$ and $\{(x,y) \mid y = \log_{1/2} x\}$

5. What is the relationship between the functions in problem 4?

6. Graph the function $\{(x,y) \mid y = a^x, a = 1\}$ and discuss why the restriction on the logarithm function, $0 < a \neq 1$, is necessary.

7. Write each of the following equations as an equivalent exponential equation.

   a. $\log_{10} 2 = y$

   b. $\log_e x = 3$

   c. $\log_e 4.3 = y$

   d. $\log_{10} x = 13.5$

e. $\log_{10} x = -3.5$ f. $\log_{10} x = 1.3249$

~~g. $\log_e x + \log_e y = \log_e z$~~ h. $\log_{10} 3 + 7 \log_{10} x = \log_{10} 10$

8. Write each of the following equations as an equivalent logaritmmic equation.

a. $10^{2.7946} = y$ b. $e^{.3978} = y$

c. $x^3 = 1000$ d. $x^1 = x$

e. $y = \log_{10} (\frac{a \cdot b}{c})$ f. $y = \log_{10} (\frac{y^3}{a})$

g. $y = \log_{10} 2 + 3 \log_{10} b - \log_{10} c$ h. $y = \log_{10} c - \frac{1}{3}\log_{10} d$

9. Solve the following equations for x.

a. $2^{x + 6} = 32$ b. $9^{2x} = 27^{3x - 4}$

c. $2^{x^2 + 4x} = \frac{1}{8}$ d. $25^{2x} = 5^{x^2 - 12}$

e. $8 = 4^{x^2} \cdot 2^{5x}$ f. $2^x \cdot 5 = 10^x$

10. Find the value of x:

a. $x = \log_3 81$ b. $\log_2 x = -5$

c. $\log_x 8 = \frac{3}{2}$ d. $\log_5 0.2 = x$

e. $\log_x 49 = 4$ f. $\log_{1/2} 8 = x$

g. $\log_9 x = -\frac{5}{2}$ h. $\log_x 0.04 = -2$

i. $\log_x \sqrt{6} = \frac{1}{4}$ j. $\log_{0.1} 10 = x$

The previous exercises, Exercise 10-5-5, contained some problems using the base e. This is the base of the natural logarithm function $\{(x,y) | y = \log_e x, x > 0\}$ sometimes written $\{(x,y) | y = \ln x, x > 0\}$. Natural logarithms are also called Napierian logarithms. The credit for the invention of the natural logarithm function, is given to John Napier who in 1617, the same year in which he died, published a book on simplification of the tedious work of multiplication titled Rabdologia. The number e is an irrational number, approximate value is 2.7182818285 and is the limit of the sequence of partial sums of the sequence

$$\{(x, \frac{1}{(x - 1)!})\}.$$

The natural and common logarithm functions are the two most widely used systems of logarithms.

10-6 Change of Base

While the bases, e and 10, of the natural and common logarithms functions are used extensively, many times it is convenient to change an expression involving logarithms into a different base. To illustrate this point we will prove the following theorem.

Theorem 10-6-1 Change of Base of Logarithms

$$\forall N,\ N > 0,\ \forall a,\ 0 < a \neq 1,\ \forall b,\ 0 < b \neq 1,\ \log_a N = \frac{\log_b N}{\log_b a}$$

proof: If we let $x = \log_a N$, then $a^x = N$.

Since $a^x = N$ then $\log_b a^x = \log_b N$

$$\log_b a^x = \log_b N \rightarrow x \log_b a = \log_b N$$

$$x = \frac{\log_b N}{\log_b a}$$

but, $x = \log_a N$

$$\therefore \log_a N = \frac{\log_b N}{\log_b a}$$

Let us first compute some logarithms using this theorem.

Example 10-6-2 Evaluate $\log_2 8$

$$\log_2 8 = \frac{\log_b 8}{\log_b 2} = \frac{\log_b 2^3}{\log_b 2} = \frac{3 \cdot \log_b 2}{\log_b 2} = 3$$

Example 10-6-3

$$\log_{25}125 = \frac{\log_b 125}{\log_b 25} = \frac{\log_b 5^3}{\log_b 5^2} = \frac{3\log_b 5}{2\log_b 5} = \frac{3}{2}$$

We may also use this theorem involving change of base in other ways.

Example 10-6-4 Solve $\log_9 N = \frac{1}{2}$ for N.

$$\log_9 N = \frac{1}{2} \rightarrow \frac{\log_b N}{\log_b 9} = \frac{1}{2}$$

$$\log_b N = \frac{1}{2}\log_b 9$$

$$\log_b N = \log_b 9^{1/2}$$

$$N = 9^{1/2}$$

$$N = 3$$

Exercise 10-6-5

Solve the following equations for x.

1. $\log_x 5 = \frac{1}{2}$
2. $\log_{27} 9 = x$
3. $\log_9 x = \frac{1}{2}$
4. $\log_{\sqrt{5}} x = -4$
5. $\log_{1/4} 64 = x$
6. $\log_x 9\sqrt{3} = 5$
7. $\log_{1/16} x = -.75$
8. $\log_x \frac{27}{8} = \frac{2}{3}$
9. Show that $\log_7 3 \cdot \log_3 7 = 1$
10. Show that $\log_a b + \log_{1/a} b = 0$
11. If $\log_x N = s$ and $\log_x b = t$, find $\log_b N$.

10-6-6 Miscellaneous Exercises

1. Write with positive exponents and simplify:

a. $\frac{a^{-1}}{b^{-1}}$ b. $(\frac{a}{b})^{-k}$, $k > 0$

c. $\frac{r}{s^{-1}}$ d. $\frac{cd^{-3}}{4a^{-2}}$

e. $\frac{x^3 \ y^3}{x^{-2} \ y^{-2}}$ f. $(d^7e^3)^{-3}$

g. $\frac{2^{-2} + b^{-2}}{a^{-2} + b^{-2}}$ h. $x^{-1} + y^{-1}$

i. $\frac{a^{-1} + b^{-1}}{(cd)^{-1}}$ j. $\frac{a^{-2} - b^{-2}}{a^{-2} + b^{-2}}$

2. Solve the following equations for x.

a. $10^{2x} - 3 = 100$ b. $e^{3x} = 16$

c. $2^{3x} = 3^{2x + 1}$ d. $5^{x + 2} = 7^{x - 2}$

e. $e^{2x} = 35$ f. $\log_3 (x + 1) + \log_3(x + 3) = 1$

g. $\log_5 \frac{3x + 4}{x} = 0$ h. $\frac{\log_{10}(7x - 12)}{\log_{10}x} = 2$

i. $\log_5 x = 3$ j. $\log_7 \frac{2x - 3}{x} = 0$

k. $\log_x 27 = \frac{3}{2}$ l. $\log_{10}x^2 - \log_{10}x = 2$

m. $e^{x \log_e b} \cdot e^{x \log_e c} = (bc)^2$

3. Find the value of x by means of the laws of exponents:

a. $3^x = 3^{-0.3}$ b. $2^{1/2} \cdot 2^{3/4} = 2^x$

c. $64^{-2/3} = 2^x$ d. $x = 16^{3/4} - 4^0 + (\frac{9}{4})^{-3/2}$

e. $\frac{4^{1/2}}{\sqrt[3]{8^2}} = 2^x$ f. $\sqrt[3]{\frac{t}{\sqrt{t}}} = t^x$

4. Find equivalent expressions to each of the following:

a. $-\log_b x$

b. $a^{\log_a}$

c. $\log_a a^x$

d. $\log_b x \cdot \log_a b$

e. $\log_a b \cdot \log_b a$

f. The inverse of $\{(x,y) | y = a^x, x > 0, 0 < a \neq 1\}$

5. Find the value of

(a) $\dfrac{\log_{25}5 - \log_{1/16}8}{\log_9 \frac{1}{27} + \log_4 1}$

(b) $\log_5 1 + \log_8 4 \cdot \sqrt[5]{16}$

(c) $\dfrac{\log_{49}7 + \log_{27}9}{\log_{1/32} 64 - \log_{3/2} \frac{4}{9}}$

(d) $\dfrac{\log_3 81 - \log_\pi 1}{\log_{2\sqrt{2}} 8 - \log_{10} 0.001}$

(e) $\dfrac{\log_{216}6^4 - \log_{27}9^{3/4}}{\log_{\sqrt{3}} 1 + \log_4 8^{-3/2}}$

(f) $\dfrac{\log_9 81^{-3/8}}{\log_{49} 7^{2/3} - \log_{64} \sqrt[5]{\frac{1}{16}}}$

10-8 Common Logarithms

When dealing with common logarithms close attention is paid to scientific notation. That is, every real number may be written as a product of an integral power of 10 and a number between 0 and 10.

Example 10-8-1

Write another name for 1129 which is a product of an integral power of 10 and a number x such that $0 \leq x \leq 10$.

$$1129 = 10^3 \cdot 1.129$$

The purpose for this discussion is that common logarithms uniquely involve this power of 10. To illustrate this point let us evaluate $\log_{10} 1129$.

Example 10-8-2

Evaluate $\log\ 1129$.

Since $1129 = 10^3 \cdot 1.129$

$$\begin{aligned}\log_{10} 1129 &= \log_{10} (10^3 \cdot 1.129)\\ &= \log_{10} 10^3 + \log_{10} 1.129\\ &= 3\log_{10} 10 + \log_{10} 1.129\\ &= 3 \cdot 1 + \log_{10} 1.129\\ &= 3 + x\end{aligned}$$

where x is the $\log_{10}$ of a number greater than or equal to one and less than 10, since $\log 1 = 0$ and $\log 10 = 1$, this means $0 \leq x < 1$.

Therefore, $\log_{10} 1129 = 3 + x$ where $0 \leq x < 1$.

Example 10-8-3

Evaluate $\log_{10} .01129$

$$.01129 = 10^{-2} \cdot 1.129$$

$$\begin{aligned}\log_{10}\ .01129 &= \log_{10}\ (10^{-2} \cdot 1.129)\\ &= \log_{10} 10^{-2} + \log_{10} 1.129\\ &= -2 + x \qquad 0 \leq x < 1\end{aligned}$$

Note that $\log_{10}$ 1.129 is needed for the evaluation of both of these logarithms.

Every logarithm to the base 10 may be written as the sum of an integer and a number x such that $0 \leq x < 1$. The integer portion of this logarithm is called the characteristic. The non-negative real number, x, is called the mantissa. Since the characteristic of a logarithm may be easily determined we only need to determine a table of mantissa. These tables are approximations and are found accurate to 3, 4, or 5 decimal places. Such tables may be seen in any traditional advanced algebra textbook.

The reason for the concern for common logarithms in the past was to simplify arithmetical computation. Since the computer is available to us at this time, we will study the properties of the general logarithm function rather than the specific common log function.

Exercise 10-8-4

Find the characteristic of the common logarithms of each of the following numbers.

1. 576
2. 15,821
3. 3
4. 0.00875
5. 0.7
6. 375.579
7. 236.7
8. 0.0002635
9. 3.15159
10. 8765
11. Construct a simple computer program which will print the number and the characteristic of its common logarithm. Use the ten numbers, problems 1-10, for DATA. Save this program and output for future use.

The Change of Base Theorem allows us the opportunity to construct a table of mantissas for common logarithms using the natural logarithm function. We simply need the equation:

$$\log N = \frac{\log_e N}{\log_e 10}$$

Exercise 10-8-5

1. Expand the program written for problem 11, Exercise 10-8-4 to include output of the mantissas of each of the numbers.
2. Use the computer and the Change of Base Theorem to construct a table of mantissas for any two digit number, x, such that $1 \leq x < 10$. That is, find mantissas for the numbers in the set $\{1.0, 1.1, 1.2, \ldots, 2.0, 2.1, 2.2, \ldots 9.0, 9.1, 9.2, \ldots 9.9\}$.

Traditionally such tables of values are constructed as in Figure 10-8-6 below.

	0	1	2	3	...	9
1	0	-	-	-	...	-
2	-	-	-	-	...	-
3	-	-	-	.518514	...	-
.						
.						
.						
9	.954242	-	-	-	...	-

Figure 10-8-6

The column of digits, 1-9 on the left side of the array is the integer portion of the number whose mantissa is being found. The top row of digits represent the possible tenths digit.

Example 10-8-7

$\text{Log}_{10}\ 9.9 = 0 + .954242$

The characteristics is 0 and $0 < \log_{10} 9.9 < 1$

It should be pointed out that the log of any number may be found using this table.

Example 10-8-8

Find the $\log_{10}$ of 3,300

$$\text{Log}_{10}\ 3{,}300 = \log_{10}(3.3 \cdot 10^3)$$

$$= \log_{10} 10^3 + \log_{10} 3.3$$

$$3 + .518514$$

If the logarithms of numbers which cannot be located directly are desired interpolation must take place just as it did when tables of trigonometric functions were used in Chapter 6.

Exercise 10-8-9

1. Sketch the graph of $y = \log_{10} x$, $x > 0$
2. Enlarge that portion of the graph in the interval $7.7 \leq x \leq 7.8$ of the domain and use similar triangles to establish the proportionalities needed to compute $\log_{10} 7.74$. Use the table constructed in Exercise 10-8-5 problem 2.
3. Use the table constructed in Exercise 10-8-5 to compute the following products and quotients.
   a. $1.3 \cdot 2.9$
   b. $1.7^2 \cdot 3.5$
   c. $(17 \cdot 8500)/(27)^2$
   d. $(340)^{7/3} \cdot 9600/124000$

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